Orthogonal sets

Definition

A set S of non-zero vectors in an inner product space is called

- orthogonal if every distinct pair of vectors in S is orthogonal.
- It is called orthonormal if it is orthogonal and every vector has length one.
- It is called an orthonormal (or orthogonal) basis if it is orthonormal (or orthogonal) and a basis.

Theorem (6.3.1)

If S is an orthogonal set of non-zero vectors in an inner product space then S is linearly independent.

Projections

Theorem (6.3.3)

If W is a finite-dimensional subspace of an inner product space V then every vector $u \in V$ can be written as

$$u = w_1 + w_2$$

where $w_1 \in W$ and $w_2 \in W^{\perp}$. In fact, this representation of u is unique.

Notation

In the previous theorem, w_1 is called the orthogonal projection of u on W and is written $proj_W(u)$.

Orthogonal projections

Theorem

If $\{v_1, v_2, \ldots, v_r\}$ is an orthogonal basis for a subspace W of an inner product space V then for any $u \in V$,

$$\begin{aligned} proj_W(u) &= \frac{\langle u, v_1\rangle}{\|v_1\|^2} v_1 + \ldots + \frac{\langle u, v_r\rangle}{\|v_r\|^2} v_r \\ &= proj_{v_1} u + \ldots + proj_{v_r} u \end{aligned} \tag{1}$$

Theorem (6.3.5)

Every non-zero finite-dimensional inner product space has an orthonormal basis.

The Gram-Schmidt process

Suppose that $\{u_1, u_2, \ldots, u_n\}$ is a basis for an inner product space V.

1. Start with u_1; call this v_1.
2. Consider u_2 and form $v_2 = u_2 - proj_{W_1} u_2$ where W_1 is the span of v_1.
3. Let W_2 be the span of v_1, v_2.
4. Consider u_3 and form $v_3 = u_3 - proj_{W_2} u_3$.
5. Let W_3 be the space spanned by v_1, v_2, v_3.
6. Repeat this process by iteratively forming v_i and W_i until $i = n$.
7. $\{v_1, v_2, \ldots, v_n\}$ forms an orthogonal basis for V. If you want an orthonormal basis, normalize each v_i.

QR-decomposition

As a consequence of the Gram-Schmidt process, one can prove:

Theorem

If A is an $m \times n$ matrix with linearly independent column vectors then one can find Q, an $m \times n$ matrix with orthonormal column vectors and R, an $n \times n$ invertible upper triangular matrix such that

$$A = QR$$

Orthogonality and complex inner product spaces

Orthogonality in complex inner product spaces is nearly identical to the real case (see discussion on pages 549 – 551). In particular,

- the definitions of orthogonal vectors, orthogonal sets, orthonormal sets and orthonormal bases are the same.
- the Pythagorean Theorem holds as do all the main theorems from section 6.3.
- the Gram-Schmidt process is still valid.