The Gram-Schmidt process

Suppose that $\{u_1, u_2, \ldots, u_n\}$ is a basis for an inner product space V.

1. Start with u_1 and "normalize" it (divide by its length so the result is of length 1); call this v_1.
2. Consider u_2 and form $u_2' = u_2 - proj_{W_1} u_2$ where W_1 is the span of v_1.
3. Now normalize u_2' and call this v_2. Let W_2 be the span of v_1, v_2.
4. Consider u_3 and form $u_3' = u_3 - proj_{W_2} u_3$.
5. Normalize u_3' and call it v_3. Let W_3 be the space spanned by v_1, v_2, v_3.
6. Repeat this process by iteratively forming v_i and W_i until $i = n$.
7. $\{v_1, v_2, \ldots, v_n\}$ forms an orthonormal basis for V.

QR-decomposition

As a consequence of the Gram-Schmidt process, one can prove:

Theorem

If A is an $m \times n$ matrix with linearly independent column vectors then one can find Q, an $m \times n$ matrix with orthonormal column vectors and R, an $n \times n$ invertible upper triangular matrix such that

$$A = QR$$

Orthogonality and complex inner product spaces

Orthogonality in complex inner product spaces is nearly identical to the real case. In particular,

- the definitions of orthogonal vectors, orthogonal sets, orthonormal sets and orthonormal bases are the same.
- the Pythagorean Theorem holds as do all the main theorems from section 6.3.
- the Gram-Schmidt process is still valid.

Projection as best approximation

Theorem

Suppose that W is a finite-dimensional subspace of an inner product space V. Then for any $u \in V$, $proj_W u$ is the closest vector in W to u; that is, if $w \in W$ is any vector other than $proj_W u$ then

$$||u - w|| > ||u - proj_W u||$$

Least squares problem

- Suppose that A is an $m \times n$ matrix and b is in R^n. The linear equations $Ax = b$ may or may not have a solution.
- The question is: find x so that Ax is closest to b.
- As x varies over all of R^n, Ax varies over the columnspace of A so we are really asking for x such that Ax equals the projection of b on the columnspace.
- In fact, we can find the necessary x by solving $A^T Ax = A^T b$. These are called the normal equations.