Continuous functions on $[0, 2\pi]$

- We have seen that $C[0, 2\pi]$ is a vector space with the integral inner product.
- With respect to this inner product, if W is a finite-dimensional subspace of $C[0, 2\pi]$, then the distance from any f to W is given by $\|f - \text{proj}_W(f)\|$.
- We have a known list of orthogonal functions in $C[0, 2\pi]$:

$$1, \sin(x), \sin(2x), \ldots \cos(x), \cos(2x), \ldots$$

Integral calculations

- If $n \neq 0$ then
$$\int_0^{2\pi} e^{int}\,dt = 0$$
and is 2π if $n = 0$.
-
$$\begin{aligned}\int_0^{2\pi} e^{int}e^{imt}\,dt &= \int_0^{2\pi}(\cos(nt)\cos(mt) - \sin(nt)\sin(mt))dt \\ &+ i\int_0^{2\pi}(\cos(nt)\sin(mt) + \sin(nt)\cos(mt))dt.\end{aligned}$$
- The imaginary part gives (substitute $-n$ for n)
$$\int_0^{2\pi}(\cos(nt)\sin(mt) + \sin(nt)\cos(mt))dt = 0$$
$$\text{and } \int_0^{2\pi}(\cos(nt)\sin(mt) - \sin(nt)\cos(mt))dt = 0.$$

Integral calculations, cont'd

- So for all m, n,

$$\int_0^{2\pi} (\cos(nt)\sin(mt) = 0$$

- If $m \neq n$ and both are positive then the real part gives (again substituting $-n$ for n)

$$\int_0^{2\pi} (\cos(nt)\cos(mt) - \sin(nt)\sin(mt))dt = 0$$

$$\text{and } \int_0^{2\pi} (\cos(nt)\cos(mt) + \sin(nt)\sin(mt))dt = 0.$$

- We conclude for $m \neq n$

$$\int_0^{2\pi} \cos(nt)\cos(mt) = 0 \text{ and } \int_0^{2\pi} \sin(nt)\sin(mt) = 0.$$

Integral calculations, cont'd

- If $m = n$ then

$$\int_0^{2\pi} (\cos^2(nt) + \sin^2(nt))dt = 2\pi$$

$$\text{and so } \int_0^{2\pi} \cos^2(nt)dt = \int_0^{2\pi} \sin^2(nt)dt = \pi.$$

- We conclude then that in $C[0, 2\pi]$ that

$$1, \sin(x), \ldots, \sin(nx), \ldots, \cos(x), \ldots, \cos(nx), \ldots$$

forms an orthogonal set and that $\|1\| = \sqrt{2\pi}$ and $\|\cos(nx)\| = \|\sin(nx)\| = \sqrt{\pi}$.

- Let W_n be the subspace generated by

$$1, \sin(x), \ldots, \sin(nx), \cos(x), \ldots, \cos(nx)$$

inside $C[0, 2\pi]$.

- Since the generators of each W_n form an orthogonal set, they are linearly independent and it is easy to compute the projection onto W_n.
- For any $f \in C[0, 2\pi]$ we compute

$$a_0 = \frac{\langle f, 1 \rangle}{\|1\|^2}, a_k = \frac{\langle f, \sin(kx) \rangle}{\|\sin(kx)\|^2} \text{ and } b_k = \frac{\langle f, \cos(kx) \rangle}{\|\cos(kx)\|^2}$$

for all $k \geq 1$.

Main Theorem

Theorem

If $f \in C[0, 2\pi]$ then $f(x)$ converges to

$$a_0 + a_1 \sin(x) + a_2 \sin(2x) + \ldots + b_1 \cos(x) + b_2 \cos(2x) + \ldots$$

with respect to $\| \cdot \|$.

Example

If W is the subspace generated by

$$1, \sin(x), \sin(2x), \ldots, \cos(x), \cos(2x), \ldots$$

then by the Main Theorem, $W^\perp = 0$. But $0^\perp$ is all of $C[0, 2\pi]$. W is not all of $C[0, 2\pi]$ since $x \notin W$ so we have an example of $(W^\perp)^\perp \neq W$.