Matrices for general linear transformations

Goal: To associate a matrices to linear transformations between finite-dimensional vector spaces.
The process: Suppose that $T : V \to W$ is a linear transformation from an n-dimensional vector space V to an m-dimensional vector space W.

1. Fix a basis B for V and B' for W.
2. Construct an $m \times n$ matrix A such that

$$A[x]_B = [T(x)]_{B'}$$

   A is called the matrix for T with respect to B and B' and we will denote it by $[T]_{B',B}$.
3. Suppose $B = \{u_1, u_2, \ldots, u_n\}$. Form A with column vectors $[T(u_1)]_{B'}, [T(u_2)]_{B'}, \ldots, [T(u_n)]_{B'}$.

Change of basis

The problem

Suppose we are given two bases

$$B = \{u_1, u_2, \ldots, u_n\} \text{ and } B' = \{u'_1, u'_2, \ldots, u'_n\}$$

for an n-dimensional vector space V; how are B and B' related?

The solution

Let P be the $n \times n$ matrix given by

$$P = ([u'_1]_B, [u'_2]_B, \ldots, [u'_n]_B)$$

Then $[v]_B = P[v]_{B'}$ for all $v \in V$. P is called the transition matrix from B' to B

Change of basis, cont'd

Theorem

If P is the transition matrix from B' to B and Q is the transition matrix from B to B' then $Q = P^{-1}$.

Change of basis, cont'd

- So the matrix representing the identity transformation, $I : V \to V$, with respect to B and B' is just the change of basis matrix P.

Theorem

If $T : V \to V$ is a linear operator on a finite-dimensional vector space V and B and B' are two bases for V then

$$[T]_{B'} = P^{-1}[T]_B P$$

where P is the change of basis matrix from B' to B.

Composition and inverse

Theorem

If $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations between finite-dimensional vector spaces and B, B' and B'' are bases for U, V and W respectively then

$$[T_2 \circ T_1]_{B'',B} = [T_2]_{B'',B'}[T_1]_{B',B}$$

Theorem

If $T : V \to V$ is a linear operator and B is a basis for V then T is one-to-one iff $[T]_B$ is invertible. If T is one-to-one then

$$[T^{-1}]_B = [T]_B^{-1}$$

Similarity

Definition

If A and B are two $n \times n$ matrices then we say A is similar to B if there is an invertible P such that $A = P^{-1}BP$.

Fact

If A and B are similar then they have the same determinant, characteristic polynomial, eigenvalues and dimensions for eigenspaces.

Determinant and eigenvalues for linear operators

Definition

- If V is a finite-dimensional vector space and T is a linear operator on V then $\det(T) = \det([T]_B)$ for any basis B of V.
- If $T : V \to V$ is a linear operator then λ is an eigenvalue for T and $v \in V, v \neq 0$ is an eigenvector if $T(v) = \lambda v$. The eigenspace associated to an eigenvalue λ is the kernel of the operator $\lambda I - T$.