Unitary matrices

Definition

An $n \times n$ complex matrix U is called unitary if $A^* = A^{-1}$.

Theorem

The following are equivalent for an $n \times n$ complex matrix A:

1. *A is unitary.*
2. *The rows of A form an orthonormal basis for C^n.*
3. *The columns of A form an orthonormal basis for C^n.*

Theorem

If A is $n \times n$ then the following are equivalent:

1. *A is unitary.*
2. $||Ax|| = ||x||$ *for all* $x \in C^n$.
3. $Ax \cdot Ay = x \cdot y$ *for all* $x, y \in C^n$.

Unitary diagonalization

Definition

Suppose A is an $n \times n$ complex matrix. Then if P diagonalizes A and P is unitary then A is said to be unitarily diagonalizable. That is, there is an unitary matrix P such that $P^{-1}AP$ is diagonal.

Theorem

If A is an $n \times n$ complex matrix then the following are equivalent:

1. *A is unitarily diagonalizable and has real eigenvalues.*
2. *A has real eigenvalues and an orthonormal set of n eigenvectors.*
3. *A is Hermitian.*

Normal matrices and Schur's Theorem

Definition

A complex $n \times n$ matrix A is called normal if $A^*A = AA^*$.

Theorem

If A is an $n \times n$ complex matrix then the following are equivalent:

1. *A is unitarily diagonalizable.*
2. *A has an orthonormal set of n eigenvectors.*
3. *A is normal.*

Theorem (Schur's theorem)

If A is any $n \times n$ complex matrix then there is an upper triangular matrix S and a unitary matrix P such that $A = P^{-1}SP$.

Cayley-Hamilton Theorem

Theorem (Cayley-Hamilton Theorem)

If A is an $n \times n$ complex matrix and $p(\lambda)$ is the characteristic polynomial of A then $p(A) = 0$.