Quadratic equations and conic sections

- A quadratic equation is one of the form

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0$$

  where at least one of a, b or c is not zero.
- There are three types of conic sections in standard position:
  - Ellipses and circles:

$$\frac{x^2}{k^2} + \frac{y^2}{l^2} = 1$$

  - Hyperbolas:

$$\frac{x^2}{k^2} - \frac{y^2}{l^2} = 1 \text{ or } \frac{y^2}{k^2} - \frac{x^2}{l^2} = 1$$

  - Parabolas:

$$y = kx^2 \text{ or } x = ky^2$$

Quadratic equations as conic sections

- Problem: How do we understand a quadratic equation as the graph of a conic section in the plane?
- Two parts of the solution:
  - The conic may not be centered at the origin: we can tell this if there is no ”cross-term” i.e. no *xy* term in the equation. Solution: complete the square to determine how translated the conic is.
  - It may be rotated. You will be able to tell this if there is a cross-term present. Solution: Orthogonally diagonalize the associated quadratic form and change variables to see what conic section you have.
  - For the quadratic equation

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0,$$

    the associated quadratic form is

$$ax^2 + 2bxy + cy^2.$$

Some questions and answers

- What are the minimum and maximum values of a given quadratic form $q(x)$ when x is restricted to the unit ball i.e. $||x|| = 1$?
- Under what circumstances will a quadratic form always be positive when $x \neq 0$?

Theorem

1. *Suppose that A is an $n \times n$ symmetric matrix with largest eigenvalue λ_1 and least eigenvalue λ_n. Then*

$$\lambda_n ||x||^2 \leq x^T A x \leq \lambda_1 ||x||^2$$

2. $x^T A x = \lambda_1 ||x||^2$ *iff x is an eigenvector for λ_1 and* $x^T A x = \lambda_n ||x||^2$ *iff x is an eigenvector for λ_n.*

Hessian form of the Second Derivative Test

Definition

For a function f of two variables with all second partial derivatives, define $H(x, y)$, the Hessian, as

$$\begin{pmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{pmatrix}.$$

Remember that a critical point for f is a point (x_0, y_0) such that $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$.

Hessian form of the Second Derivative Test

Theorem

Suppose that (x_0, y_0) is a critical point of $f(x, y)$ and that f has continuous second derivatives in an open neighbourhood of (x_0, y_0). Then if $H(x_0, y_0)$ is the Hessian of f at (x_0, y_0) then

1. *f has a relative minimum at (x_0, y_0) if $H(x_0, y_0)$ is positive definite.*
2. *f has a relative maximum at (x_0, y_0) if $H(x_0, y_0)$ is negative definite.*
3. *f has a saddle point at (x_0, y_0) if $H(x_0, y_0)$ is indefinite.*

Singular value decomposition

Theorem

Suppose that A is an $m \times n$ complex matrix with rank k. Then there are unitary matrices U and V as well as positive numbers $\mu_1, \ldots, \mu_k$ such that A can be written as

$$V \left(\begin{array}{cccc|ccc} \mu_1 & & & & 0 & \ldots & 0 \\ & \mu_2 & & & \vdots & & \vdots \\ & & \ddots & & \vdots & & \vdots \\ & & & \mu_k & 0 & \ldots & 0 \\ \hline 0 & \ldots & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & \ldots & \ldots & 0 & 0 & \ldots & 0 \end{array}\right) U.$$

Jordan blocks

Definition

A square complex matrix of the following form

$$\begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & & \lambda & 1 \\ 0 & \dots & & 0 & \lambda \end{pmatrix}$$

is called a Jordan block (for λ).

Definition

A square complex matrix of the following form

$$\begin{pmatrix} J_1 & & & & \\ & J_2 & & & \\ & & J_3 & & \\ & & & \ddots & \\ & & & & J_k \end{pmatrix}$$

where $J_1, J_2, \ldots J_k$ are Jordan blocks of various sizes, is said to be in Jordan canonical form.

The main theorem

Theorem

- *Every square complex matrix is similar to one in Jordan canonical form.*
- *In fact, up to permutation of the Jordan blocks, the Jordan canonical form is unique.*
- *The similarity class of a given $n \times n$ complex matrix A is determined by the following data: for each eigenvalue λ of A and each $k \leq n$, the number of $k \times k$ Jordan blocks for λ appearing in the canonical form for A.*
- *If λ is an eigenvalue of A, the dimension of the eigenspace for λ is the number of Jordan blocks for λ in the Jordan canonical form of A.*