Linear independence

Definition

If $S = \{v_1, v_2, \ldots, v_r\}$ is a non-empty set of vectors such that the only solution for scalars $k_1, k_2, \ldots, k_r$ of the equation

$$k_1 v_1 + k_2 v_2 + \ldots + k_r v_r = 0$$

is $k_1 = k_2 = \ldots = k_r = 0$ then S is said to be linearly independent. Otherwise, S is linearly dependent.

Basis and Dimension

Definition

If V is a vector space and $S = \{v_1, v_2 \ldots, v_n\}$ is a set of vectors in V then S is said to be a basis for V if

1. S is linearly independent and
2. S spans V.

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Definition

A vector space V is called finite-dimensional if it has a finite basis. Otherwise it is called infinite-dimensional.

Theorem (4.5.1)

If V is a finite-dimensional vector space then all bases for V have the same number of vectors.

Complex vector spaces

- Suppose V is a set together with the operations $+$ and multiplication by complex numbers i.e. the scalars are now complex. Then we call V a complex vector space if the same 10 axioms from section 4.1 are satisfied.
- The definition of subspace remains the same for complex vector spaces; the main Theorem for identifying subspaces is also the same i.e. it is sufficient for a subset of a vector space to be closed under $+$ and scalar multiplication to be a subspace.

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- Some things do change:

Definition

If $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ are vectors in C^n then we define the dot product as

$$u \cdot v = u_1 \bar{v}_1 + u_2 \bar{v}_2 + \ldots + u_n \bar{v}_n$$

Properties of the dot product

Theorem (5.3.1)

If u, v and w are vectors in C^n and k is any complex number (scalar) then

1. $u \cdot v = \overline{v \cdot u}$,
2. $(u + v) \cdot w = u \cdot w + v \cdot w$,
3. $(ku) \cdot v = k(u \cdot v)$, *and*
4. $u \cdot u \geq 0$. *Moreover* $u \cdot u = 0$ *iff* $u = 0$.

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The complex norm

For $u = (u_1, u_2, \ldots, u_n)$ in C^n, we define

$$||u|| = \sqrt{u \cdot u} = \sqrt{|u_1|^2 + |u_2|^2 + \ldots + |u_n|^2}$$

Linear independence and bases in complex vector spaces

- Linear independence in complex vector spaces is identical to linear independence in real vector spaces with the only change being that the scalars are complex.
- A basis for a complex vector space is a maximal linearly independent subset of that space.
- Every complex vector space has a basis and the size of the basis is determined by the space itself so in particular if the space is finite-dimensional then all bases have the same size.

How to form a basis

Theorem (Plus/Minus Theorem, 4.5.3)

Let S be a non-empty subset of a vector space V.

1. *If S is linearly independent and v is in V but not in the span of S then $S \cup \{v\}$ is linearly independent.*
2. *If v in S is expressible as a linear combination of other vectors from S then the spans of S and $S \setminus \{v\}$ (S without v) are the same.*