From Friday

Theorem

If A is an n x n real or complex matrix, $v_1, \ldots, v_k$ are eigenvectors for A which correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ then $v_1, \ldots, v_k$ are linearly independent.

1. In order to prove this we assumed that $v_1, \ldots, v_k$ were linearly *dependent* and that in fact k was the smallest possible under these circumstances.
2. We then wrote that for some $c_1, \ldots, c_k$ not all zero,

$$c_1 v_1 + \ldots + c_k v_k = 0.$$

I should have said this means that *all* $c_i \neq 0$ by the minimality of k.

From Friday

Then

$$A(c_1 v_1 + \ldots + c_k v_k) = A0 = 0$$

so

$$c_1 \lambda_1 v_1 + \ldots + c_k \lambda_k v_k = 0$$

and by multiplying by λ_1,

$$c_1 \lambda_1 v_1 + \ldots + c_k \lambda_1 v_k = 0$$

so by subtracting we get

$$c_2(\lambda_2 - \lambda_1) v_2 + \ldots + c_k(\lambda_k - \lambda_1) v_k = 0$$

which contracts the minimality of k.

Diagonalizability

Definition

A square matrix A is called diagonalizable if there is an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. P is said to diagonalize A.

Theorem (5.2.1)

The following are equivalent for an $n \times n$ matrix A:

1. *A is diagonalizable.*
2. *A has n linearly independent eignevectors.*

Inner product spaces

Definition

An inner product on a real vector space V is a function that associates a real number $\langle u, v\rangle$ to each pair of vectors $u, v \in V$ such that the following axioms are satisfied, for every u, v and w in V and any scalar k:

1. $\langle u, v\rangle = \langle v, u\rangle$,
2. $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$,
3. $\langle ku, v\rangle = k\langle u, v\rangle$, and
4. $\langle u, u\rangle \geq 0$. Moreover $\langle u, u\rangle = 0$ iff $u = 0$.

V together with an inner product is called an inner product space.

Norm and distance

Definition

If V is an inner product space then the norm of a vector $v \in V$ is written $||v||$ and defined as

$$||v|| = \sqrt{\langle v, v \rangle}$$

For $u, v \in V$, the distance between u and v is written $d(u, v)$ and is defined as

$$d(u, v) = ||u - v||$$

Properties of inner products

Theorem (6.1.1)

If u, v and w are vectors in a real inner product space and k is any scalar then

1. $\langle 0, v \rangle = 0$
2. $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$
3. $\langle u, kv \rangle = k\langle u, v \rangle$
4. $\langle u - v, w \rangle = \langle u, w \rangle - \langle v, w \rangle$
5. $\langle u, v - w \rangle = \langle u, v \rangle - \langle u, w \rangle$

Complex inner product spaces

Definition

An inner product on a complex vector space V is a function that associates a complex number $\langle u, v\rangle$ to each pair of vectors $u, v \in V$ such that the following axioms are satisfied, for every u, v and w in V, and scalar k:

1. $\langle u, v\rangle = \overline{\langle v, u\rangle}$,
2. $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$,
3. $\langle (ku), v\rangle = k\langle u, v\rangle$, and
4. $\langle u, u\rangle \geq 0$. Moreover $\langle u, u\rangle = 0$ iff $u = 0$.

V together with an inner product is called a complex inner product space.

The definition of norm and distance in a complex inner product space is the same as in the real case.