Inner product spaces

Definition

An inner product on a real vector space V is a function that associates a real number $\langle u, v \rangle$ to each pair of vectors $u, v \in V$ such that the following axioms are satisfied, for every u, v and w in V and any scalar k:

1. $\langle u, v \rangle = \langle v, u \rangle$,
2. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$,
3. $\langle (ku), v \rangle = k\langle u, v \rangle$, and
4. $\langle u, u \rangle \geq 0$. Moreover $\langle u, u \rangle = 0$ iff $u = 0$.

V together with an inner product is called an inner product space.

Norm and distance

Definition

If V is an inner product space then the norm of a vector $v \in V$ is written $||v||$ and defined as

$$||v|| = \sqrt{\langle v, v \rangle}$$

For $u, v \in V$, the distance between u and v is written $d(u, v)$ and is defined as

$$d(u, v) = ||u - v||$$

Properties of inner products

Theorem (6.1.1)

If u, v and w are vectors in a real inner product space and k is any scalar then

1. $\langle 0, v\rangle = 0$
2. $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$
3. $\langle u, kv\rangle = k\langle u, v\rangle$
4. $\langle u - v, w\rangle = \langle u, w\rangle - \langle v, w\rangle$
5. $\langle u, v - w\rangle = \langle u, v\rangle - \langle u, w\rangle$

Cauchy-Schwarz inequality

Theorem

If u and v are vectors in an inner product space then

$$|\langle u, v\rangle| \leq ||u||||v||$$

Length and Distance

Theorem

If u and v are vectors in an inner product space and k is any scalar then:

1. $||u|| \geq 0$
2. $||u|| = 0$ *iff* $u = 0$
3. $||ku|| = |k|||u||$
4. *(triangle inequality)* $||u + v|| \leq ||u|| + ||v||$

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Theorem

If u, v and w are vectors in an inner product space then:

1. $d(u, v) \geq 0$
2. $d(u, v) = 0$ *iff* $u = v$
3. $d(u, v) = d(v, u)$
4. *(triangle inequality)* $d(u, w) \leq d(u, v) + d(v, w)$

Angles and orthogonality

Definition

If u and v are vectors in an inner product space then

1. we define the angle θ between them to be that θ such that $0 \leq \theta \leq \pi$ and
$$cos(\theta) = \frac{\langle u, v \rangle}{||u||||v||}$$
2. we say that u and v are orthogonal if this angle is $\pi/2$; that is, we say that u and v are orthogonal if $\langle u, v \rangle = 0$.

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Theorem (Pythagorean Theorem)

In an inner product space, if u and v are orthogonal then

$$||u + v||^2 = ||u||^2 + ||v||^2$$