Mathematics 2R3 Practice Test 2

Drs. Hart and van Tuyl — Winter, 2019

Last Name: Solutions — Initials: ________

Student No.: ____________________

- The test is 50 minutes long.
- The test has 6 pages and 5 questions and is printed on BOTH sides of the paper.
- You are responsible for ensuring that your copy of the paper is complete. Bring any discrepancies to the attention of the invigilator.
- Attempt all questions and write your answers in the space provided.
- Marks are indicated next to each question; the total number of marks is 25.
- You may use a McMaster standard Casio fx-991 calculator (no communication capability); no other aids are not permitted.
- Use pen to write your test. If you use a pencil, your test will not be accepted for regrading (if needed).

Good Luck!

Score

Question	1	2	3	4	5	Total
Points	5	5	5	5	5	25
Score						

continued ...

1. (5 marks) Put your answer in the space provided for each part.

(a) The range of a linear transformation is a vector space. True or False.

If $T: V \to W$ is a linear transf, $R(T) \subseteq W$ is a subspace, so $R(T)$ is a vector space

TRUE

(b) If V is an n-dimensional vector space and $T : V \to V$ is a linear operator with range V then T is one-to-one. True or False.

By rank-nullity, $\dim \mathrm{Nul}(T) + \dim R(T) = \dim V$.
We are given $\dim R(T) = \dim V$ (since $R(T) = V$).
So $\dim \mathrm{Nul}(T) = 0$. So $\ker(T) = \{0\} \iff T$ ~~1-1~~ one-to-one

TRUE

(c) The real vector spaces of 2×2 real matrices and polynomials of degree at most 3 with real coefficients are isomorphic. True or False.

$\dim M_{2,2}(\mathbb{R}) = 4$
$\dim \mathbb{P}_3 = 4$.
So $M_{2,2}(\mathbb{R})$ and $\mathbb{P}_3$ both isomorphic to $\mathbb{R}^4$, so isomorphic to each other

TRUE

(d) Suppose that A is an $n \times n$ matrix and for every $x \in \mathbf{R}^n$, $T(x) = x^T A x$ then T is a linear transformation. True or False.

Here's a counter example. Let $n = 2$ and $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Then $T((x_1, x_2)) = [x_1\ x_2] A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1\ x_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + x_2^2$

FALSE

Then $T((1,0)+(1,0)) = T((2,0)) = 4$ $\neq$
But $T((1,0)) + T((1,0)) = 1+1 = 2$

(e) Suppose that $T : V \to W$ is a surjective linear transformation, the dimension of W is 4 and the dimension of V is 7. What is the dimension of the kernel of T?

By rank-nullity,
$\dim \ker(T) + \dim R(T) = \dim V$
Now $\dim V = 7$ (given) and since $R(T) = W$ (because T surjective) $\dim R(T) = \dim W = 4$.
So $\dim \ker(T) + 4 = 7 \iff \dim \ker(T) = 3$

3

2. (5 marks) Consider the inner product space C[−1, 1] with the inner product given by

$$\langle f, g\rangle = \int_{-1}^{1} fg\,dx.$$

(a) (3 marks) Apply the Gram-Schmidt process to the linearly independent set $\{1, x, x^2\}$ in C[−1, 1] to obtain an orthogonal set.

$u_1 = 1$

$u_2 = x - \frac{\langle x, 1\rangle}{\|1\|^2} 1$ $\quad \langle x, 1\rangle = \int_{-1}^{1} x\,dx = 0$

$= x$ $\quad \langle x^2, 1\rangle = \int_{-1}^{1} x^2\,dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3}$

$u_3 = x^2 - \frac{\langle x^2, 1\rangle}{\|1\|^2} 1 - \frac{\langle x^2, x\rangle}{\|x\|^2} x$ $\quad \langle x^2, x\rangle = \int_{-1}^{1} x^3\,dx = 0$

$\|1\|^2 = \langle 1, 1\rangle = \int_{-1}^{1} 1\,dx = 2$

$= x^2 - \frac{2/3}{2} 1 - 0$ $\quad \|x\|^2 = \langle x, x\rangle = \int_{-1}^{1} x^2\,dx = 2/3$

So $\{1, x, x^2 - \frac{1}{3}\}$ is an orthogonal set.

(b) (2 marks) If W is the subspace generated by $\{1, x, x^2\}$, determine the projection of x^3 onto W.

We can compute $\text{proj}_W x^3$ if we use an orthogonal basis for W (as found in part (a))

$$\text{proj}_W x^3 = \frac{\langle x^3, 1\rangle}{\|1\|^2} 1 + \frac{\langle x^3, x\rangle}{\|x\|^2} x + \frac{\langle x^3, x^2 - \frac{1}{3}\rangle}{\|x^2 - \frac{1}{3}\|^2}\left(x^2 - \frac{1}{3}\right)$$

We compute the needed integrals

$\langle x^3, 1\rangle = \int_{-1}^{1} x^3\,dx = 0 \qquad \langle x^3, x\rangle = \int_{-1}^{1} x^4\,dx = \frac{x^5}{5}\Big|_{-1}^{1} = \frac{2}{5}$

$\langle x^3, x^2 - \frac{1}{3}\rangle = \int_{-1}^{1}\left(x^5 - \frac{x^3}{3}\right)dx = \frac{x^6}{6} - \frac{x^4}{12}\Big|_{-1}^{1} = \left(\frac{1}{6} - \frac{1}{12}\right) - \left(\frac{1}{6} - \frac{1}{12}\right) = 0$

Need $\|x\|^2 = \langle x, x\rangle = \int_{-1}^{1} x^2\,dx = \frac{2}{3}$.

So $\boxed{\text{proj}_W x^3 = \frac{2/5}{2/3} x = \frac{3}{5} x}$

3. P_2 is the real vector space of polynomials of degree less than or equal to 2. Define $T : P_2 \to P_2$ by $T(p(x)) = p(x-1)$.

   (a) (3 marks) Show that T is a linear transformation.

Check two properties of a linear transformation

- Let $p(x), g(x) \in \mathbb{P}_2$ So $p(x) = a_0 + a_1 x + a_2 x^2$ and $g(x) = b_0 + b_1 x + b_2 x^2$

Then $T(p(x)) + T(g(x)) = [a_0 + a_1(x-1) + a_2(x-1)^2] + [b_0 + b_1(x-1) + b_2(x-1)^2]$

$= [a_0 + b_0] + (a_1+b_1)(x-1) + (a_2+b_2)[x-1]^2$

$= T(\cancel{p(x)}(a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2)$

$= T(p(x) + g(x))$.

- Let $p(x) \in P(x)$ and $k \in \mathbb{R}$. Then

$T(kp(x)) = T(ka_0 + ka_1 x + ka_2 x^2) = ka_0 + ka_1(x-1) + ka_2(x-1)^2$

$= k[a_0 + a_1(x-1) + a_2(x-1)^2] = k\,T(p(x))$.

So T is a linear transf.

   (b) (2 marks) Determine the range and kernel of T

$\ker(T) = \{p(x) = a_0 + a_1 x + a_2 x^2 \mid T(p(x)) = a_0 + a_1(x-1) + a_2(x-1)^2 = 0\}$

<u>Note</u> $a_0 + a_1(x-1) + a_2(x-1)^2 = a_0 + a_1 x - a_1 + a_2 x^2 - 2a_2 x + a_2 = 0$

$$\Leftrightarrow \left.\begin{array}{r} (a_0 - a_1 + a_2) = 0 \\ a_1 - 2a_2 = 0 \\ a_2 = 0 \end{array}\right\} \Rightarrow a_0 = a_1 = a_2 = 0$$

So $\boxed{\ker(T) = \{0\}.}$

Note that $\ker(T) = \{0\}$, So T is injective.

So $\dim R(T) = \dim P_2$. But this implies

$$\boxed{R(T) = P_2}$$

continued ...

4. Suppose that P_2 is the vector space of real polynomials of degree ≤ 2 and $T : P_2 \to \mathbf{R}^3$ is a linear transformation satisfying

$$T(1) = (1, 0, 3), T(x + 1) = (2, -3, 0) \text{ and } T(x^2 + x + 1) = (1, 0, 1).$$

(a) (2 marks) Compute $T(1 - x + x^2)$.

Observe $1 - x + x^2 = (x^2+x+1) - 2(x+1) + 2\cdot 1$.

So $T(1-x+x^2) = T\left((x^2+x+1) - 2(x+1) + 2\cdot 1\right)$

$= T(x^2+x+1) - 2T(x+1) + 2T(1)$

$= (1,0,1) - 2(2,-3,0) + 2(1,0,3)$

$\boxed{= (-1, 6, 7)}$

(you need to write $a_0+a_1x+a_2x^2$ in terms of $1, x+1, x^2+x+1$.

(b) (3 marks) Write out a formula for $T(a_0 + a_1x + a_2x^2)$.

Observe $a_0 + a_1x + a_2x^2 = a_2(x^2+x+1) - (a_2 - a_1)(x+1) + (a_0 - a_1)\cdot 1$

So $T(a_0 + a_1x + a_2x^2) = a_2 T(x^2+x+1) - (a_2-a_1)T(x+1) + (a_0-a_1)T(1)$

$= a_2(1,0,1) - (a_2-a_1)(2,-3,0) + (a_0-a_1)(1,0,3)$

$= (a_2 - 2a_2 + 2a_1 + a_0 - a_1, +3a_2 - 3a_1, a_2 + 3a_0 - 3a_1)$

$= (-a_2 + a_1 + a_0, 3a_2 - 3a_1, a_2 + 3a_0 - 3a_1)$

Note In (a), $a_0 = 1$, $a_1 = -1$, $a_2 = 1$. When we plug this in, we get $(-1-1+1, 3\cdot 1 - 3(-1), 1 + 3\cdot 1 - 3(-1)) = (-1, 6, 7)$

continued . . .

5. Prove that if $T : V \to W$ is a one-to-one linear transformation then T^{-1} is a linear transformation from $R(T)$ to V.

Recall T^{-1} is only defined on $R(T)$, ie

$T^{-1}: R(T) \to V$ is given by

$T^{-1}(\vec{w}) = \vec{x}$ when $T(\vec{x}) = \vec{w}$.

Proof that T^{-1} is a linear transformation:

- Let $\vec{w}_1, \vec{w}_2 \in R(T)$ with $\vec{x}_1, \vec{x}_2 \in V$ such that $T(\vec{x}_1) = \vec{w}_1$ & $T(\vec{x}_2) = \vec{w}_2$.

Then

$= T^{-1}(\vec{w}_1) + T^{-1}(\vec{w}_2) = \vec{x}_1 + \vec{x}_2$.

But since $T(\vec{x}_1) + T(\vec{x}_2) = T(\vec{x}_1 + \vec{x}_2) = \vec{w}_1 + \vec{w}_2$, we have

$T^{-1}(\vec{w}_1 + \vec{w}_2) = \vec{x}_1 + \vec{x}_2$.

Putting pieces together, we have $T^{-1}(\vec{w}_1) + T^{-1}(\vec{w}_2) = T^{-1}(\vec{w}_1 + \vec{w}_2)$.

- Let $\vec{w} \in R(T)$ and $c \in \mathbb{R}$. Let $\vec{x} \in V$ be such that $T(\vec{x}) = \vec{w}$.

Since T is a linear transf, $T(c\vec{x}) = cT(\vec{x}) = c\vec{w}$.

So $T^{-1}(c\vec{w}) = c\vec{x} = c\,T^{-1}(\vec{w})$. (since $T^{-1}(\vec{w}) = \vec{x}$)

THE END