Mathematics 2R3 Practice Test 3

Drs. Hart and van Tuyl

Last Name: SOLUTIONS Initials:________

Student No.:____________________

- The test is 50 minutes long.
- The test has 6 pages and 5 questions and is printed on BOTH sides of the paper.
- You are responsible for ensuring that your copy of the paper is complete. Bring any discrepancies to the attention of the invigilator.
- Attempt all questions and write your answers in the space provided.
- Marks are indicated next to each question; the total number of marks is 25.
- You may use a McMaster standard Casio fx-991 calculator (no communication capability); no other aids are not permitted.
- Use pen to write your test. If you use a pencil, your test will not be accepted for regrading (if needed).

Good Luck!

Score

Question	1	2	3	4	5	Total
Points	5	5	5	5	5	25
Score						

continued ...

Note: In the test, you don't need to provide justifications. They are provided here to help you study

1. (5 marks) Answer the following true or false and put your answer in the space provided.

   (a) If A is an $n \times n$ real symmetric matrix then any two eigenvectors are orthogonal.

   FALSE

   Look @ Ex 1 on page 412. The ~~use~~ eigenvectors of $\lambda = 2$ are not orthogonal. However, you can find a basis where they are orthogonal. It is true, however, that if $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors of distinct eigenvalues, they are orthogonal

   (b) If A and B are similar then they have the same characteristic polynomial.

   TRUE

   This is mentioned in the table on page 303 and again in the table on page 485

   (c) The eigenvalues of an Hermitian matrix are real.

   TRUE

   This is Thm 7.5.2(a)

   (d) If T is an invertible linear transformation then its kernel is the zero subspace.

   TRUE

   T must be one-to-one to be invertible, so $\ker(T) = \{0\}$

   (e) A matrix cannot be similar to itself.

   FALSE

   For any matrix A,

   $A = I_n A I_n^{-1}$

2. (5 points) Suppose that $D : P^2 \to P^2$ is the differentiation operator $D(p) = p'$.

(a) Find the matrix associated to D relative to the basis $B = \{1, 1+x, 1+x+x^2\}$.

By def$^{\underline{n}}$, $[D]_{B,B} = [\,[D(1)]_B \; [D(1+x)]_B \; [D(1+x+x^2)]_B\,]$

Now $D(1) = 1' = 0 = 0\cdot 1 + 0\cdot(1+x) + 0(1+x+x^2) \Rightarrow [D(1)]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$D(1+x) = (1+x)' = 1 = 1\cdot 1 + 0\cdot(1+x) + 0(1+x+x^2) \Rightarrow [D(1+x)]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$D(1+x+x^2) = (1+x+x^2)' = 1+2x = (-1)(1) + 2(1+x) + 0(1+x+x^2)$

$\Rightarrow [D(1+x+x^2)]_B = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

So $\boxed{[D]_{B,B} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}}$ ← sol$^{\underline{n}}$

(b) Use the matrix from part (a) to compute $D(3+2x+x^2)$.

First find $[3+2x+x^2]_B$ since $1\cdot 1 + 1(1+x) + 1(1+x+x^2) = 3+2x+x^2$,

$[3+2x+x^2]_B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

sol$^{\underline{n}}$

So $[D(3+2x+x^2)]_B = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \Rightarrow D(3+2x+x^2) = 2(1+x) = 2+2x}$

(Note that this is correct since $D(3+2x+x^2) = (3+2x+x^2)'$
$= 2+2x$

continued . . .

3. Suppose that $T : \mathbf{R}^3 \to \mathbf{R}^3$ is given by

$$T(x, y, z) = (x_1 - x_2, x_2 - x_3, x_3 - x_1).$$

(a) (2 points) Compute the determinant of T.

For (a) & (b), we need $[T]_{\mathcal{E},\mathcal{E}}$ (we will use the standard basis)

$$[T]_{\mathcal{E}\mathcal{E}} = \left[[T(e_1)]_{\mathcal{E}} \; [T(e_2)]_{\mathcal{E}} \; [T(e_3)]_{\mathcal{E}} \right] = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

So $\det(T) = \det\left(\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}\right) = 1 + (-1) + 0 - 0 - 0 - 0 = \boxed{0 = \det(T)}$

(b) (3 points) Determine the eigenvalues of T.

Find eigenvalues of T & note we directly know 0 is one of the eigenvalues since $\det(T) = 0$

$$\lambda I_3 - [T]_{\mathcal{E}} = \begin{bmatrix} \lambda-1 & 1 & 0 \\ 0 & \lambda-1 & 1 \\ 1 & 0 & \lambda-1 \end{bmatrix} \Rightarrow \text{char}([T]_{\mathcal{E}}) = (\lambda-1)^3 + 1$$

$$= \lambda^3 - 3\lambda^2 + 3\lambda - 1 + 1$$

$$= \lambda^3 - 3\lambda^2 + 3\lambda$$

$$= \lambda(\lambda^2 - 3\lambda + 3) = 0$$

$$\lambda = 0 \text{ or } \lambda = \frac{3 \pm \sqrt{9 - 4(3)}}{2} = \frac{3 \pm \sqrt{-3}}{2} = \frac{3 \pm \sqrt{3}i}{2}$$

So eigenvalues $\boxed{\lambda = 0, \frac{3+\sqrt{3}i}{2}, \frac{3-\sqrt{3}i}{2}}$

continued ...

4. (5 points) Find a unitary matrix P which unitarily diagonalizes A and determine $P^{-1}AP$ where

$$A = \begin{pmatrix} 3 & i \\ -i & 3 \end{pmatrix}.$$

Step 1: Find eigenvalues and eigenvectors

$$\lambda I_2 - A = \begin{pmatrix} \lambda-3 & -i \\ i & \lambda-3 \end{pmatrix} \Rightarrow \text{char}(A) = (\lambda-3)^2 - (-i^2) = (\lambda-3)^2 - 1$$

$$= \lambda^2 - 6\lambda + 9 - 1 = \lambda^2 - 6\lambda + 8$$

$$= (\lambda-2)(\lambda-4) = 0$$

Eigenvalues $\lambda = 2, \lambda = 4$

$\boxed{\lambda = 2}$ $\begin{bmatrix} 2-3 & -i \\ i & 2-3 \end{bmatrix} = \begin{bmatrix} -1 & -i \\ i & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$ x_2 free, $x_1 = -i x_2$

So Eigenspace $\left\{ x_2 \begin{bmatrix} -i \\ 1 \end{bmatrix} \middle| x_2 \cancel{\text{free}} \in \mathbb{C} \right\}$

$\boxed{\lambda = 4}$ $\begin{bmatrix} 4-3 & -i \\ i & 4-3 \end{bmatrix} = \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$ x_2 free, $x_1 = i x_2$

So Eigenspace $\left\{ x_2 \begin{bmatrix} i \\ 1 \end{bmatrix} \middle| x_2 \in \mathbb{C} \right\}$

Step 2 Find orthonormal basis for each eigenspace

$$\begin{bmatrix} -i \\ 1 \end{bmatrix} \Rightarrow \frac{1}{\sqrt{(-i)(i)+1}} \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} -i/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} i \\ 1 \end{bmatrix} \Rightarrow \frac{1}{\sqrt{i(-i)+1}} \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} i/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Note: there are two choices for the columns since you can also take the reverse direction

sol^n

Step 3 Put pieces together

$$\boxed{A = \begin{bmatrix} -i/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & i/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} i/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -i/\sqrt{2} \end{bmatrix} = P D P^*}$$

continued . . .

5. (5 points) Suppose that A is a normal matrix. We know there is a unitary matrix U and a diagonal matrix D such that $A = U^*DU$.

(a) Suppose that λ is an eigenvalue for A and x is an eigenvector for λ. Show that Ux is an eigenvector for D.

Given $A\vec{x} = \lambda\vec{x}$. and $UA = DU$

Then $(UA)(\vec{x}) = (DU)(\vec{x}) \Rightarrow U(A\vec{x}) = D(U\vec{x}) \Rightarrow U(\lambda\vec{x}) = D(U\vec{x})$

$\Rightarrow \boxed{D(U\vec{x}) = \lambda(U\vec{x})}$ ← this expression shows that $U\vec{x}$ is an eigenvector of D (Note that λ is the eigenvalue of $U\vec{x}$)

(b) Assume as a fact that if x and y are eigenvectors for different eigenvalues of a diagonal matrix then $x \cdot y = 0$. Use this fact and part (a) to conclude that for a normal matrix, eigenvectors for distinct eigenvalues are orthogonal.

Let $\vec{x}$ and $\vec{y}$ be distinct eigenvectors of distinct eigenvalues of a normal matrix A.

By part (a) $U\vec{x}$ and $U\vec{y}$ are eigenvectors of distinct eigenvalues of a diagonal matrix D. By the fact, we have

$(U\vec{x})\cdot(U\vec{y}) = 0.$

Because U is unitary, $(U\vec{v})\cdot(U\vec{w}) = \vec{v}\cdot\vec{w}$ for all $\vec{v}, \vec{w} \in \mathbb{C}^n$. So, for $\vec{x}, \vec{y}$,

$\boxed{\vec{x}\cdot\vec{y} = (U\vec{x})\cdot(U\vec{y}) = \vec{0}.}$

This shows the eigenvectors are orthogonal.

THE END