Assignment 1, Math 2S3
Due January 20, in class

1. In class we showed that in any group, the inverse is unique. If $(G, \cdot)$ is a group and we write $-u$ for the inverse of u, show that $-(-u) = u$ for all $u \in G$.

2. Give an example of a non-commutative semi-group which is not a monoid.

3. (a) Show that if $F \subseteq K$ are both fields and addition and multiplication on F are the restrictions of addition and multiplication on K (F is a subfield of K) then K can be thought of as an F-vector space with addition just the addition on K and scalar multiplication by elements of F simply multiplication by elements of F.

   (b) What is the dimension of $Q(i)$ as a Q-vector space? What is the dimension of C as an R-vector space? Is the dimension of R as a Q-vector space finite? You can use the fact that there is no bijection between Q^n and R for any $n \in N$.

4. Let's build a finite field which is not Z_p for any prime p. Start with the polynomial $p(x) = x^2 + x + 1$ over Z_2 and show that it is irreducible over Z_2. Now consider the set $Z_2(t) = \{a + bt : a, b \in Z_2\}$ where t is supposed to be thought of as a "solution" of $p(x) = 0$. Define $+$ and $\cdot$ by

$$(a + bt) + (c + dt) = (a + c) + (b + d)t$$

and

$$(a + bt) \cdot (c + dt) = (ac + bd) + (ad + bc + bd)t$$

Show that $Z_2[t]$ is a field of size 4 and that t solves p in this field.