Solutions to Test 1

1. a) $(V,+)$ is an abelian group if V is non-empty, $+: V\times V \to V$ and $+$ is associative, has an identity, inverses and is commutative.

b) 2×2 invertible matrices ~~which are~~ form a non-abelian group under matrix multiplication. $\circ$ is associative, I is the identity and the matrices are invertible.

$$\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix} \neq \begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}.$$

2. a) If V, W are K-vector spaces then $f: V\to W$ is a linear map if
   1) for all $u,v\in V$, $f(u+v) = f(u)+f(v)$.
   2) for all $\lambda\in K$, $u\in V$, $f(\lambda u) = \lambda f(u)$.

b) In all vector spaces, $0\cdot u = 0$ so if $f: V\to W$ is a linear map and $v\in V$ then

$$f(0) = f(0\cdot v) = 0 f(v) = 0.$$

3. We need to show that $Im(f)$ is closed under $+$ and scalar mult. If $w_1, w_2 \in Im(f)$ then there are $v_1, v_2 \in V$ so that $f(v_1) = w_1$ and $f(v_2) = w_2$.

Then $w_1 + w_2 = f(v_1) + f(v_2) = f(v_1+v_2) \in Im(f)$.

If $\lambda \in K$ then $\lambda w_1 = \lambda f(v_1) = f(\lambda v_1) \in Im(f)$ so $Im(f)$ is a subspace of W.

4. Choose a basis $v_1 \cdots v_m$ for W. Since $v_1 \cdots v_m$ are lin. indep. in V, we can extend this set to a basis for V, say $v_1 \cdots v_m, v_{m+1} \cdots v_n$.

But then $\dim(V) = n \geq m = \dim(W)$.

5. It suffices to find a basis for the space of all symmetric $n \times n$ matrices.

Let $A_{ii} = \begin{pmatrix} 0 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & \ddots \; 0 \end{pmatrix}$ i^{th} spot on the diagonal.

If $i<j$ then $A_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ij^{th} spot. ji^{th} spot.

A_{ij} is symmetric when $i \leq j$ and if $A = (a_{ij})$ is symmetric

then $A = \sum_i a_{ii} A_{ii} + \sum_{i<j} a_{ij} A_{ij}$ since $a_{ij} = a_{ji}$ for all i,j.

If $\sum_i \lambda_{ii} A_{ii} + \sum_{i<j} \lambda_{ij} A_{ij} = 0$ then we have $\lambda_{ij} = 0$ for all $i \leq j$ so

$\{A_{ij} : i \leq j\}$ forms a basis and it has size $\frac{n(n+1)}{2}$.