Solutions to Assignment #3

1. We need to show that $I = \{ f \in K[x] : f(A) = 0 \}$ is closed under + and mult. by any $g \in K[x]$.

If $f_1, f_2 \in I$ then $(f_1 + f_2)(A) = f_1(A) + f_2(A) = 0$ so $f_1 + f_2 \in I$. If $g \in K[x]$ and $f \in I$ then

$$(gf)(A) = g(A)f(A) = g(A) \cdot 0 = 0 \quad \text{so } gf \in I.$$

So I is an ideal. Since all ideals in $K[x]$ are singly generated, $I = \langle f \rangle$ for some f.

f has the least degree of any poly. in I and since $f(A) = 0$, f is a minimal poly. for A.

If g is the char. poly of A then $g \in I$ so f divides g.

2. The only eigenvalue of a nilpotent matrix is 0 (If $Nv = \lambda v$ then $N^r v = \lambda^r v = 0$ so $\lambda = 0$). So the characteristic poly. of an $n \times n$ nilpotent matrix is x^n.

3. Suppose first of all that A is upper triangular. Write $A = D + N$ where D is diagonal and N is upper triangular with only 0 on the diagonal.

So $N = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$. Claim: $N^n = 0$.

To check this notice that $Ne_i \in \langle e_1, \cdots e_{i-1} \rangle$ for $i > 1$ and $Ne_1 = 0$.

If we apply N n times, we get $N^n e_i = 0$ for all i. So $N^n = 0$.

Now if A is not upper triangular, we know it is similar to an upper triangular matrix so choose invertible P so that

$P^{-1} A P = D + N$ with D a diagonal matrix and N nilpotent. Then $A = PDP^{-1} + PNP^{-1}$ and PDP^{-1} is diagonalizable and still PNP^{-1} is nilpotent.

$$(PNP^{-1})^n = PN^n P^{-1} = 0.$$

4. The zero $n \times n$ matrix has minimal poly X for any n.

$$N = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$$ ← a single diagonal of 1's above the diagonal.

Characteristic poly of N is x^n. For $r < n$,

$N^r e_n = e_{n-r} \neq 0$ so $N^r \neq 0$ for any $r < n$.

The minimal poly then is x^n.