Solutions to Assignment #4

1. Suppose that $g = qf + r$. If $h = \gcd(f,g)$ then h divides both g and f so h divides $g - qf$ i.e. h divides r. So h divides g and r so h divides the $\gcd(g,r)$. In the other direction if $h' = \gcd(g,r)$ then h' divides $qg + r$ i.e. h' divides f so h' divides h. This shows $h = h'$.

To determine the gcd of g, f write:

$$g = q_1 f + r_1 \quad , \quad \deg(r_1) < \deg(f)$$

$$f = q_2 r_1 + r_2 \quad , \quad \deg(r_2) < \deg(r_1)$$

$$\vdots$$

$$r_{n} = q_{n+1} r_{n+1} + r_n \quad \text{until } r_n = 0 \text{ at which}$$

point, $\gcd(f,g) = r_{n+1}$.

2. A is similar to a matrix B in upper triangular form, say $B = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ and $A = C^{-1}BC$ for some invertible C.

$f(A) = C^{-1} f(B) C$ and the diagonal of $f(B)$ is $f(\lambda_1), \ldots, f(\lambda_n)$. So the char. poly. of $f(A)$ is

$$\prod_{i=1}^{m} (x - f(\lambda_i))^{k_i}$$

3. By the Jordan canonical form theorem, every such matrix ~~has~~ is similar to one with Jordan blocks on the diagonal; the sizes must add to 6 so we could have:

1 block - 6
2 blocks - 5+1, 4+2, 3+3
3 blocks - 4+1+1, 3+2+1, 2+2+2
4 blocks - 3+1+1+1, 2+2+1+1
5 blocks - 2+1+1+1+1
6 blocks - 1+1+1+1+1+1

The ~~size~~ dimension of the eigenspace for 0 is equal to the number of blocks so we just have to see that the cases with 2, 3 and 4 blocks are all dissimilar.
The degree of the minimal poly. is the size of the largest block so this separates all the cases with 2, 3 or 4 blocks.

4. We need to show that det is multilinear, alternating and $\det(I) = 1$.

Multilinear: Suppose $A = (a_{ij})$ and the i^{th} column

$$\begin{pmatrix} a_{i1} \\ \vdots \\ a_{in} \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} + \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

Then $\det(A) = \sum_{i_1 \cdots i_n} (-1)^{N(i_1 \cdots i_n)} a_{1i_1} \cdots a_{ni_n}$

$= \sum_{i_1 \cdots i_n} (-1)^{N(i_1 \cdots i_n)} a_{1i_1} \cdots (b_i + c_i) \cdots a_{ni_n}$

substituted wherever $i_j = i$

$= \sum_{i_1 \cdots i_n} (-1)^{N(i_1 \cdots i_n)} a_{1i_1} \cdots b_i \cdots a_{ni_n} + \sum_{i_1 \cdots i_n} (-1)^{N(i_1 \cdots i_n)} a_{1i_1} \cdots c_i \cdots a_{ni_n}$

$= \det(B) + \det(C)$

where B is A with $\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ in the i^{th} column and C is

A with $\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ in the i^{th} column.

If B is the same as A with the i^{th} column $\bar{a}_i$ replaced by $\lambda \bar{a}_i$ then a factor of λ can be pulled out of the summation so $\det(B) = \lambda \det(A)$.

Alternating: Suppose $A = (\bar{a}_1 \cdots \bar{a}_{i-1}\ \bar{u}\ \bar{u}\ \bar{a}_{i+2} \cdots \bar{a}_n)$.

We want to show $\det(A) = 0$.

Consider any term in the summation:

$(-1)^{N(i_1 \cdots i_n)} a_{1i_1} \cdots a_{1i_n}$ The column corresponding to i is some i_j and to $i+1$ is some i_k. This same product appears when i_j and i_k are swapped with coefficient $(-1)^{N(i_1 \cdots \overset{\curvearrowleft}{i_k\, i_j} \cdots i_n)}$.

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So how does the inversion number change when we exchange two numbers? It changes by -1 (by induction on how far they are apart).

So the products are the same but one has $+1$ in front and one has -1 so they sum to 0. This is true for all pairs from the summation so

$$\det(A) = 0$$

$\det(I) = 1$: For $\det(I)$, notice that $a_{1i_1} \cdots a_{ni_n} = 0$ unless $i_1 = 1, \cdots i_n = n$ at which point we get

$$\det(I) = 1.$$