Assignment 1, Math 3EE3

Due Jan. 20 in class

(1) Suppose R is a ring. We say that $p \in R$ is a projection if $p^2 = p$. Show that if p is a projection then $pRp = \{pap : a \in R\}$ is a subring of R for which p is a multiplicative identity. Moreover, show that if S is a subring of R and S has a multiplicative identity p then p is a projection and $S \subseteq pRp$.

(2) Suppose that R is a ring with $+$ and $\cdot$. Fix any set X and let R^X be the set of all functions from X to R. Define $+$ and $\cdot$ on R^X as follows: for $f, g \in R^X$, $f+g$ and fg are the functions satisfying for all $x \in X$

$$(f+g)(x) = f(x) + g(x) \text{ and } (fg)(x) = f(x)g(x)$$

Show that R^X is a ring.

(3) Let's give two proofs that if R is a ring and $X \subseteq R$ then there is a minimal subring of R which contains X.

(a) Consider the set $\{S \subseteq R : X \subseteq S \text{ and } S \text{ is a subring}\}$. Consider the intersection of all these subrings is also a subring of R and it is the smallest subring containing X.

(b) Suppose $x_1, \ldots, x_n \in X$; call $x_1 x_2 \ldots x_n$, the product of these elements, a word from X. Let S be the set of all finite sums and differences of words from X. That is, if $w_1, \ldots, w_n$ and $u_1, \ldots, u_m$ are words from X then

$$(w_1 + \ldots + w_n) - (u_1 + \ldots + u_m)$$

is in S. Show that S is the smallest subring in R containing X; use the convention that the empty sum is 0.

(4) Show that if R is a ring then $M_n(R)$ is also a ring where addition is defined by $(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$ and multiplication is given by $(a_{ij})(b_{ij}) = (c_{ij}$ where $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$. In order to show that multiplication is associative, consider $A \in M_n(R)$ to be a function from R^n to R^n by the usual multiplication of matrices and vectors. Then argue that matrix multiplication is just composition of functions.

(5) Consider the following 3 complex 2×2 matrices

$$i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Let H be defined as the subset of 2×2 matrices of the form $aI + bi + cj + dk$ where a, b, c and d are real numbers. Prove that H is a division ring as a subring of $M_2(C)$ and is non-commutative. This ring is called the quaternions.