Assignment 4, Math 3EE3
Due Mar. 12 in class

(1) Suppose that p is prime and $\sigma : Z[x] \to Z_p[x]$ is defined by computing the coefficients from a polynomial in $Z[x]$ mod p.
  (a) Prove that σ is a homomorphism.
  (b) Show that if $f \in Z[x]$ and $\sigma(f)$ have the same degree then if $\sigma(f)$ is irreducible over Z_p then f is irreducible over Z.
  (c) Use this to show that $x^3 + 17x + 36$ is irreducible over Z.

(2) In class we showed that any field F can be extended to one in which all polynomials over F have a solution. Here is another proof of that fact: Let P be the set of non-constant polynomials over a field F and let X be the set of finite subsets of P. For each $\Delta \in X$ we can find an extension of F, F_Δ such that every polynomial in Δ has a solution in F_Δ. Let $R = \prod_{\Delta \in X} F_\Delta$ and let I be

$$\{\bar{a} \in R : \text{ for some } \Delta \in X, \text{ if } \Delta \subseteq \Sigma \in X \text{ then } a_\Sigma = 0\}$$

Here we are considering $\bar{a} \in R$ as the sequence $\langle a_\Sigma : \Sigma \in X \rangle$.
  (a) Show that I is a proper ideal. Choose a maximal ideal J with $I \subseteq J$ and let $K = R/J$; K is a field.
  (b) Let $\Phi : F \to K$ be defined by $\Phi(a) = \langle a : \Delta \in X \rangle / J$ i.e. the constant sequence a modulo J. Show that Φ is an embedding. Hence we can associate F with the image of Φ.
  (c) Show that K satisfies all polynomials over F.

(3) Suppose that F is a field, $S \subseteq F^n$ and I is an ideal in $F[x_1, \ldots, x_n] = F[\bar{x}]$. Define

$$I(S) = \{f \in F[\bar{x}] : f(\bar{s}) = 0 \text{ for all } \bar{s} \in S\}$$

and

$$V(I) = \{\bar{s} \in F^n : f(\bar{s}) = 0 \text{ for all } f \in I\}$$

  (a) Prove that $I(S)$ is an ideal.
  (b) Prove that $S \subseteq V(I(S))$.
  (c) Give an example of a subset S of R^2 for which $S \neq V(I(S))$.