Assignment 5, Math 3EE3
Due Apr. 2, in class

1. Here is a problem that involves Zorn's Lemma and algebraic closures: we will prove that if F is a field then any two algebraic closures (algebraically closed fields which are algebraic over F) are isomorphic by an isomorphism that is the identity on F. To start, suppose that K_1 and K_2 are algebraic closures of F.

   (a) Let P be the set of partial functions f from K_1 to K_2 with the following properties:

      i. F is contained in the domain of f and f restricted to F is the identity on F.

      ii. f is a field isomorphism between its domain and range.

   Order P as follows $f \leq g$ if the domain of f is contained in the domain of g and for $a \in \mathrm{dom}(f)$, $f(a) = g(a)$. Prove that Zorn's Lemma applies to this partial order.

   (b) By Zorn's Lemma, choose a maximal f in P. You need to prove two things:

      i. The domain of f is K_1.

      ii. The range of f is K_2.

   Hint: the two cases are similar. In the first case, imagine $a \in K_1$ which is not in the domain of f. Use the fact that K_2 is algebraically closed to extend f to include a in its domain.

2. Show that if $f \in Z_p[x]$ is irreducible then f divides $x^{p^n} - x$ for some $n \in N$.

3. Determine the minimal polynomials for $\cos(2\pi/5)$ and $\cos(2\pi/7)$ and discuss the significance of the answer to the constructibility of a regular pentagon and heptagon.