Solutions to Test 1

1 a) $\mathbb{Z}_6$

b) $\mathbb{Z}$

c) the quaterions.

d) $2\mathbb{Z}$

2 a) Eisenstein, $p=3$ - irreducible.

b) reducible, $x=1$ is a root

c) $x^4-2 = (x^2-\sqrt{2})(x^2+\sqrt{2})$, reducible.

3. R/I is a commutative ring with 1; we need to see that if $a/I \neq 0$ then it has a mult. inverse. $a/I \neq 0$ means $a \notin I$ so the ideal generated by I and a, J properly contains I so $J=R$ since I is maximal! Since $1 \in R$, $1 = ra+s$ for some $r \in R$, $s \in I$. Then $(r+I)(a+I) = 1+I$ so $r+I$ is the inverse of $a+I$.

4. a) $\mathbb{Z}_{17}$ is a field so $\{0\}$ and $\mathbb{Z}_{17}$ are the only idede.

b) $\mathbb{Z}_{16}$: $\{0\}$, $2\mathbb{Z}_{16}$, $4\mathbb{Z}_{16}$, $8\mathbb{Z}_{16}$ and $\mathbb{Z}_{16}$

c) $M_2(\mathbb{Z}_2)$: $\{0\}$ and $M_2(\mathbb{Z}_2)$ - ~~to~~ see this let's show that if $R = M_n(F)$, any n and field F, the only ideals are $\{0\}$ and R.

If $I \subseteq R$ is an ideal and $A \in I$, $A \neq 0$ then pick some i,j s.t. $a_{ij} \neq 0$. If e_{kl} is the matrix with 0 everywhere except a 1 in the k,l spot.

Then $\frac{1}{a_{ij}} e_{ik} A e_{kj} = e_{kk}$ for any k (calculate)

and so $e_{kk} \in I$ for all k. $I_n = \sum_k e_{kk}$ so if $I \neq \{0\}$ then $I = R$.

5. $\mathbb{Z}[\sqrt{3}] \subseteq \mathbb{Q}[\sqrt{3}]$ and $\mathbb{Q}[\sqrt{3}]$ is a field. In order to see that it is the fraction field, it suffices to see that $\mathbb{Q}[\sqrt{3}]$ is minimal as a field containing $\mathbb{Z}[\sqrt{3}]$. To see this, it is enough to see that every element of $\mathbb{Q}[\sqrt{3}]$ is a quotient of elements from $\mathbb{Z}[\sqrt{3}]$. So if $a+b\sqrt{3} \in \mathbb{Q}[\sqrt{3}]$,

$a = \frac{p}{q}$ and $b = \frac{r}{q}$ (taking common denominator) where

$p,q,r \in \mathbb{Z}$ and $q \neq 0$. So $a+b\sqrt{3} = \frac{p+r\sqrt{3}}{q}$.

and $p+r\sqrt{3}, q \in \mathbb{Z}[\sqrt{3}]$.