Assignment 1 Solutions

1. To see that pRp is a subring, we need to see that it is closed under $+, \cdot$ and $-$ from R.

Closure under $+$: $pap + pbp = p(a+b)p$ for all $a, b \in R$.

Closure under $\cdot$: $(pap)(pbp) = p(ap^2b)p$ for all $a, b \in R$.

Closure under $-$: $-(pap) = p(-a)p$ for all $a \in R$.

To see that p is the mult. identity for pRp, notice $p(pap) = p^2ap = pap$ and $(pap)p = pap^2 = pap$ for all $a \in R$.

Now suppose $S \subseteq R$ has a mult. identity p. Then $p^2 = p$ so p is a projection in R. and for any $a \in S$, $pap = a$ (since p is the identity in S) so $a \in pRp$.

2. $(R, +)$ is an abelian group and R^X as defined is the same as the definition of R^X as an abelian group so $(R^X, +)$ is an abelian group.

$\cdot$ is associative since for every $x \in X$ and $f, g, h \in R^X$ $f(x)(g(x)h(x)) = (f(x)g(x))h(x)$ by assoc. in R.

Distributivity holds similarly since it holds in R when you evaluate at any $x \in X$.

3 a) First of all, this set is not empty since $X \subseteq R$ and R is a subring of itself.

Now $S_{int} = \bigcap \{ S \subseteq R : X \subseteq S, S \text{ is a subring} \}$ then we need to see that S_{int} is closed under $+$, $\cdot$ and $-$.

Closure ~~under +~~ : If $a, b \in S_{int}$ then $a, b \in S$ for every S mentioned in the set on the RHS. But S is a subring so $a+b$, ab and $-a \in S$. Since this is true for all S, $a+b$, ab and $-a \in S_{int}$.

b) This is a little longer than a) but also more constructive: With S as described in the question we need to see that S is closed under $+$, $\cdot$ and $-$.

$+$: If we have two expressions $(w_1 + \cdots + w_n) - (u_1 + \cdots + u_m)$ and $(v_1 + \cdots + v_k) - (y_1 + \cdots + y_l)$ where all of w_i, u_i, v_i and y_i are words in X then

$$((w_1 + \cdots + w_n) - (u_1 + \cdots + u_m)) + ((v_1 + \cdots + v_k) - (y_1 + \cdots + y_l))$$

$$= (w_1 + \cdots + v_k) - (u_1 + \cdots + y_l) \text{ which is in } S.$$

$\cdot$: With expressions as above

$$((w_1 + \cdots + w_n) - (u_1 + \cdots + u_m))((v_1 + \cdots + v_k) - (y_1 + \cdots + y_l))$$

$$= (w_1 v_1 + w_1 v_2 + \cdots + w_n v_k) + (w_1 y_1 + \cdots + w_n y_l)$$

$$-((u_1 v_1 + \cdots u_m v_k) + (w_1 y_1 + \cdots + w_n y_l))$$

The product of two words from X is also a word from S so this is in S.

$-$: $-((w_1 + \cdots + w_n) - (u_1 + \cdots + u_m))$

$= (u_1 + \cdots + u_m) - (w_1 + \cdots + w_n)$

So S is a subring of R and since every subring of R containing X must contain every word from X and every sum and difference of words, S is the smallest such subring.

4. Checking that $(M_n(R), +)$ is an abelian group amounts to noticing that $(R, +)$ is an abelian group in each ij entry.

The checking of associativity would have been easier if we assumed that R had a 1 ~~but so it goes~~ so let's assume that.

For $A \in M_n(R)$, define the function $f_A : R^n \to R^n$

by $f_A(\bar{x}) = A\bar{x} = (a_{ij})\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \left(\sum_{j=1}^{n} a_{ij} x_j \right)$

It is easy to see that f_A is ~~linear~~ an additive homomorphism i.e. $f_A(\bar{x} + \bar{y}) = f_A(\bar{x}) + f_A(\bar{y})$ and

so f_A is determined by its values on all elements of $\mathbb{R}^n$ of the form $e_j = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{pmatrix} \leftarrow j^{th}$ spot.

The i^{th} entry of $f_A(e_j)$ is a_{ij} and so the i^{th} entry of $f_B(f_A(e_j)) = \sum_{k=1}^{n} b_{ik} a_{kj}$

If we compute the i^{th} entry of $f_{BA}(e_j)$ we also get $\sum_{k=1}^{n} b_{ik} a_{kj}$. So $f_B \circ f_A = f_{BA}$. This shows associativity immediately since composition of functions is associative and we would have $f_C \circ (f_B \circ f_A) = f_{CBA} = (f_C \circ f_B) \circ f_A$ which shows

$C(BA) = (CB)A$ for all $A, B, C \in M_n(\mathbb{R})$.

We can do a similar calculation for distributivity based on the fact that $f_A + f_B = f_{A+B}$. This gives, for any C

$f_C \circ (f_A + f_B) = f_{C(A+B)} = f_{CA} + f_{CB}$ so $C(A+B) = CA + CB$

and $(f_A + f_B) \circ f_C = f_{(A+B)C} = f_{AC} + f_{AC}$ so $(A+B)C = AC + BC$.

We could have proved both associativity and distributivity directly. Suppose $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$

Then $BA = \left(\sum_{k=1}^{n} b_{ik} a_{kj} \right)$ and $C(BA) = \left(\sum_{k=1}^{n} c_{ik} \left(\sum_{l=1}^{n} b_{kl} a_{lj} \right) \right)$

$$= \left(\sum_{k,l=1}^{n} c_{ik} b_{kl} a_{lj} \right)$$

Likewise $CB = \left(\sum_{k=1}^{n} c_{ik} b_{kj} \right)$ and

$$(CB)A = \left(\sum_{k=1}^{n} \left(\sum_{l=1}^{n} c_{il} b_{lk} \right) a_{kj} \right)$$

$$= \left(\sum_{k,l=1}^{n} c_{il} b_{lk} a_{kj} \right)$$

So $C(BA) = (CB)A$ - associativity

— . —

For distributivity, with the same notation

$$(A+B)C = \left(\sum_{k=1}^{n} (a_{ik} + b_{ik}) c_{kj} \right)$$

$$= \left(\sum_{k=1}^{n} a_{ik} c_{kj} + \sum_{k=1}^{n} b_{ik} c_{kj} \right)$$

$$= AC + BC$$

and similar for $A(B+C) = AB + AC$.

#5. We first show that H is closed under $+$ and $\cdot$. ($-I \in H$ so we don't need to show closure under $-$) This will show that H is a ring with unit.

One checks that $ij = k$, $jk = -i$ and $ki = j$ and by using distributivity, we get H closed under $\cdot$; closure under $+$ is clear.

So we must show that if $x = aI + bi + cj + dk \neq 0$ i.e. at least one of a, b, c or $d \neq 0$ then x is invertible.

But $x = \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$ and so $\det(x) = a^2+b^2+c^2+d^2 \neq 0$

x is invertible in $M_2(\mathbb{C})$ but is it invertible in H?

$$x^{-1} = \frac{1}{\det(x)} \begin{pmatrix} a-bi & -c-di \\ c-di & a+bi \end{pmatrix} = \frac{1}{\det(x)} \left(aI - bi - cj - dk \right)$$

which is in H. So H is a division ring;

$ij = -ji = k$ so H is not commutative.