Solutions to Assignment #4

1. Suppose that $f, g \in \mathbb{Z}[x]$. We need to see that $\sigma(f+g) = \sigma(f) + \sigma(g)$ and $\sigma(fg) = \sigma(f)\sigma(g)$.

Let $\varphi : \mathbb{Z} \to \mathbb{Z}_p$ sending $a \mapsto \bar{a}$.

Then if $f = a_n x^n + \cdots + a_0$, $g = b_n x^n + \cdots + b_0$ (we can assume we have coeff. up to x^n by setting some b's or a's to 0).

$$\sigma(f+g) = \varphi(a_n + b_n)x^n + \cdots + \varphi(a_0 + b_0)$$

$$= \varphi(a_n)x^n + \varphi(b_n)x^n + \cdots + \varphi(a_0) + \varphi(b_0)$$

$$= \sigma(f) + \sigma(g).$$

For $\sigma(fg)$, notice that the coeff. of x^m is

$\sum_{i+j=m} a_i b_j$ so the coeff. of x^m in $\sigma(fg)$ is

$$\varphi\left(\sum_{i+j=m} a_i b_j\right) = \sum_{i+j=m} \varphi(a_i b_j) = \sum_{i+j=m} \varphi(a_i)\varphi(b_j)$$

which is the coeff of x^m for $\sigma(f)\sigma(g)$ so $\sigma(fg) = \sigma(f)\sigma(g)$.

b) Suppose that f is ~~i~~reducible, say $f = gh$. But then $\sigma(f) = \sigma(g)\sigma(h)$ and since the degree of f and $\sigma(f)$ are the same, we also must have the degrees of $\sigma(g), \sigma(h)$ are not 0 so $\sigma(f)$ reducible.

c) Consider $p = 5$. Then $\sigma(x^3 + 17x + 36) = x^3 + 2x + 1$ over $\mathbb{Z}_5$. But $x^3 + 2x + 1$ has no root in $\mathbb{Z}_5$ (check!) and it is of degree 3 so it is irreducible. Hence $x^3 + 17x + 36$ is irreducible over $\mathbb{Z}$. Notice that Eisenstein does not apply here.

2. a) To see that I is an ideal, suppose that $\bar{a} = \langle a_\Sigma : \Sigma \in X \rangle$ and $\bar{b} = \langle b_\Sigma : \Sigma \in X \rangle$ are both in I. Then there is some $\Delta, \Phi \in X$ s.t. if $\Delta \subseteq \Sigma$ then $a_\Sigma = 0$ and if $\Phi \subseteq \Sigma$ then $b_\Sigma = 0$. But $\Delta \cup \Phi \in X$ and if $\Delta \cup \Phi \subseteq \Sigma$ then $a_\Sigma + b_\Sigma = 0$ so $\bar{a} + \bar{b} \in I$.

If $\bar{r} = \langle r_\Sigma : \Sigma \in X \rangle$ and $\bar{a} \overset{\in I}{\text{is}}$ as above then for some $\Delta \in X$, if $\Delta \subseteq \Sigma$ then $a_\Sigma = 0$. But then if $\Delta \subseteq \Sigma$, $r_\Sigma a_\Sigma = 0$ so $\bar{r}\bar{a} \in I$.

I is proper since the constant sequence 1 is not in the ideal.

b) As we just noted, 1 is not in any proper ideal so $1 \notin J$ which implies Φ is an embedding since F is a field.

c) Suppose $f \in F[x]$. If $f \in \Delta$ then f has a solution in F_Δ; call this a_Δ. Let $\bar{a} \in R$ be defined to be a_Δ if $f \in \Delta$ and 0 otherwise.

Compute $f(\bar{a}) + J$: If $f \in \Delta$ then $f(a_\Delta) = 0$ so $f(\bar{a}) \in I \subseteq J$ so $f(\bar{a}) + J = 0$ in K.

[Note: This construction is known as an ultraproduct construction and works in much wider generality.]

3. a) Suppose that $f, g \in I(S)$. Then for any $\bar{s} \in S$

$f(\bar{s}) + g(\bar{s}) = 0$ so $f+g \in I(S)$.

If $f \in I(S)$ and $g \in F[x_1, \ldots x_n]$ and $\bar{s} \in S$ then

$(gf)(\bar{s}) = g(\bar{s}) f(\bar{s}) = 0$ so $gf \in I(S)$.

b) Suppose $\bar{s} \in S$ and $f \in I(S)$. Then by definition, $f(\bar{s}) = 0$ so $\bar{s} \in V(I(S))$.

c) There are several ways to construct a counter-example. Notice that $V(I)$ for an ideal I is a closed set (it is in fact the intersection of finitely many zero sets of polynomials by HBT).

So if S is open, say the open unit ball then $S \neq V(I(S))$.

S could also be closed and fail to equal $V(I(S))$; for instance, if S was the closed unit ball then the only polynomials in two variables which is 0 on the entire unit ball is the 0 polynomial so in this case $V(I(S)) = \mathbb{R}^2$.