

# Solutions to Assignment #1

①

1. Consider the set  $S$  of strings of propositional symbols for which  $N(s) = (\# \text{ of } \neg \text{'s} - \# \text{ of } \rightarrow \text{'s})$  in  $s$  is  $\geq 0$ . We show that  $S$  is closed under the three conditions in the definition of formula:

①  $N(p) = 0$  where  $p$  is any prop. var. so  $p \in S$ .

②  $N(\neg\phi) = N(\phi) \geq 0$  for any  $\phi \in S$ .

③  $N(\phi \square \psi) = N(\phi) + N(\psi) + 1 \geq 0$   
if  $\square$  is either  $\vee$  or  $\wedge$

$$N(\phi \rightarrow \psi) = N(\phi) + N(\psi) \geq 0.$$

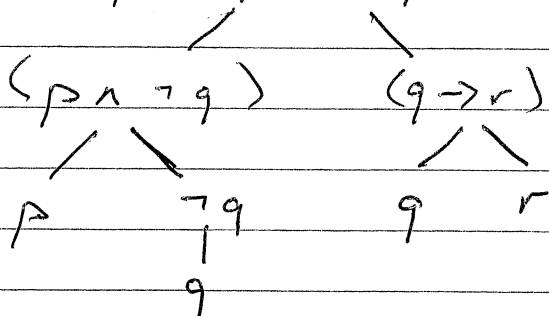
In either case, if  $\phi, \psi \in S$  then  $\phi \square \psi \in S$  for any binary connective.

So we conclude  $N(\phi) \geq 0$  for any formula  $\phi$ .

$$2. \phi_1 := (\neg((p \wedge \neg q) \vee (q \rightarrow r)) \rightarrow s)$$

$$\phi_2 := \neg((p \wedge \neg q) \vee (q \rightarrow r)) \quad s$$

$$\phi_3 := ((p \wedge \neg q) \vee (q \rightarrow r))$$



So the subformulas are  $p, q, r, \neg q, (p \wedge \neg q), s, (q \rightarrow r), \phi_3, \phi_2$  and  $\phi_1$ .

(2)

3. We need to show the set of accepted strings satisfies:

- ① every prop. var is accepted.
- ② if  $s$  is accepted then  $\neg s$  is accepted and
- ③ if  $s, t$  are accepted then  $(s \square t)$  is accepted for any binary connective  $\square$ .

By induction on formulas, if we do these 3 things then every formula is accepted.

① is just (b) of the algorithm.

② is (d).

So we need to verify ③. We need to prove one small lemma that was alluded to in class:

Lemma: If  $\phi$  is an <sup>accepted string</sup> ~~formula~~ then the bracket count of any binary connective is  $> 0$ .

Pf/ We do this by induction on <sup>the length of the string</sup> ~~formulas~~ ~~or well~~. For propositional variables ~~and truth~~, there are no binary connectives. For  $\neg \phi$ , the bracket count for a binary connective doesn't change. If we are looking at  $(\phi \square \phi)$  then the bracket count of a binary connective in  $\phi$  or  $\phi$  increases by 1 so remains  $> 0$ . For  $\square$ , the bracket count is  $1 > 0$ .  $\square$

Now back to ③: Suppose we have  $(s \square t)$

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For accepted strings  $s, t$  and binary connective  $\square$ .

By our lemma, the bracket count of any binary connective in  $s$  or  $t$  is  $> 0$ . So the bracket count of that binary connective in  $(s \square t)$  is  $> 1$ . Since  $s$  is accepted, the bracket count of  $s$  is allowable so the bracket count of  $\square$  is 1. Putting this together then when the algorithm sees  $(s \square t)$ , the first binary connective with bracket count 1 is  $\square$  and  $s$  and  $t$  are accepted so it accepts  $(s \square t)$ .