Solutions to Assignment 2

1. To show that every formula is equivalent to one in conjunctive normal form, do the following:

Suppose the formula φ involves the propositional variables $p_1, \cdots, p_n$. Consider any truth assignment v for $p_1, \cdots p_n$. We look for a formula ψ_v which makes ψ_v false on assignment v but true otherwise. Consider

$$\sigma_i = \begin{cases} \neg p_i & \text{if } v(p_i) = T \\ p_i & \text{if } v(p_i) = F \end{cases}$$

Then $\bigvee_{i=1}^{n} \sigma_i = \psi_v$ is false at v but true for all other truth assignments. If the formula φ is ~~true~~ false for truth assignments $v_1, \cdots v_k$ then consider

$\psi = \bigwedge_{i=1}^{k} \psi_{v_i}$ which is false at $v_1 \cdots v_k$ and true otherwise, ψ is in CNF and is equivalent to φ.

2. a) We showed in class that $\varphi, \neg\varphi \vdash \psi$ for any formulas φ, ψ. Consider then $\neg\neg\varphi, \neg\varphi \vdash \neg A$ for some axiom A. By deduction we have

$\neg\neg\varphi \vdash (\neg\varphi \to \neg A)$ and $(\neg\varphi \to \neg A) \to (A \to \varphi)$ is an axiom. So $\neg\neg\varphi \vdash (A \to \varphi)$ by M.P. and since A is an axiom, $\neg\neg\varphi \vdash \varphi$.

b) $\varphi \to ((\varphi \to \varphi) \to \varphi)$ axiom 1

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$\varphi \to (\varphi \to \varphi) \to (\varphi \to ((\varphi \to \varphi) \to \varphi) \to (\varphi \to \varphi))$ axiom 2.

$\varphi \to ((\varphi \to \varphi) \to \varphi) \to (\varphi \to \varphi)$ M.P.

$\varphi \to \varphi$ M.P.

3. Suppose we have a proof of $\Gamma, \varphi \vdash \psi$ and we use the proof from class to convert it to a proof $\Gamma \vdash (\varphi \to \psi)$.

There are three possibilities:

① ψ is in Γ or is an axiom. To get $\varphi \to \psi$ then we have the derivation ψ, $\psi \to (\varphi \to \psi)$ and $\varphi \to \psi$ by M.P. So the entry ψ in the proof from Γ, φ is replaced by 3 formulas.

② ψ is φ. This is 2(b) above and the proof is 5 lines. We have to do this at most once in the proof.

③ We obtain ψ by M.P. from $\theta \to \psi$, θ earlier in the proof. We assume we have $\varphi \to \theta$ and $\varphi \to (\theta \to \psi)$ from Γ already. We need one instance of axiom 2 and 2 uses of M.P. so 3 lines replaces 1.

So if the proof $\Gamma, \varphi \vdash \psi$ is ~~at~~ has n lines then $\Gamma \vdash \varphi \to \psi$ has at most $3n+2$ lines —

3 lines each for possibilities 1 and 3 and 3+2 lines for case 2 but only once.

4. a) $\Rightarrow$ b) Suppose b) is false. That is, suppose $\Gamma \models \varphi$ but $\Gamma_0 \not\models \varphi$ for all finite $\Gamma_0 \subseteq \Gamma$.

Look at $\Gamma \cup \{\neg\varphi\}$. Every finite subset of this set is satisfiable (if Σ is finite and $\Sigma \subseteq \Gamma \cup \{\neg\varphi\}$ then certainly $\Sigma \cup \{\neg\varphi\}$ is finite and satisfiable by assumption. So Σ is satisfiable.). By a) then $\Gamma \cup \{\neg\varphi\}$ is satisfiable contradicting $\Gamma \models \varphi$.

b) $\Rightarrow$ a) Suppose a) is false. So every finite subset of Γ is satisfiable but Γ is not. Then $\Gamma \models \perp$ since Γ cannot be satisfied. By b), there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \models \perp$. But Γ_0 can be satisfied which is a contradiction.

We know now by the completeness theorem that b) is actually true. It is difficult to see this directly from the definition because although φ may involve only finitely many propositional variables, we may not be able to cleverly pick out $\Gamma_0 \subseteq \Gamma$ which only involves those variables.