Solutions to Assignment 5

1. We show that every formula is equivalent to a quantifier free formula modulo the theory of infinite sets.

The only terms are variables and the only relation is $=$ so the only atomic formulas look like $x_i = x_j$ for variables x_i and x_j.

Now every quantifier-free formula with free variables $x_1 \cdots x_n$ has the form $\bigvee_{i \in I} \bigwedge_{j \in J_i} \varphi_{ij}$ where

each φ_{ij} is of the form either $x_i = x_j$ or $x_i \neq x_j$.

Using the same logic as with dense linear orders, since for every i, j we have $(x_i = x_j \vee x_i \neq x_j)$ is always true, we can, after rearranging the DNF, assume that every conjunct contains exactly one of $x_i = x_j$ or $x_i \neq x_j$ for every $i, j \leq n$. Call such a conjunct complete.

Define a relation $i \sim j$ iff $x_i = x_j$ is present in the conjunct. If $\sim$ is not an equivalence relation then the conjunct is equivalent to false and we can ignore it.

So each quantifier free formula looks like

$\bigvee_{i \in I} \bigwedge_{j \in J_i} \varphi_{ij}$ where the φ_{ij}'s are $\pm$ atomic formulas, each conjunct is complete and the $\sim$ relation on the conjunct is an equivalence relation.

Now we look at arbitrary formulas by induction: The only case we need to consider is one that looks like.

$$\exists x_k \bigvee_{i \in I} \bigwedge_{j \in J_i} \varphi_{ij}$$ with the DNF as above.

This is logically equivalent to $\bigvee_{i \in I} \exists x_k \bigwedge_{j \in J_i} \varphi_{ij}$

So we are done if we see that $\exists x_k \bigwedge_{j \in J} \varphi_j$ is equivalent to a quantifier-free formula where the conjunct is complete and $\sim$ is an equivalence relation. If $x_i = x_k$ for $i \neq k$ is present then the truth of $\bigwedge_{j \in J} \varphi_i$ is equivalent to the formula where you substitute x_i for x_k uniformly. This is quantifier-free. If $x_k \neq x_i$ is present for every i then the truth of this formula in the theory of infinite sets depends only on the conjuncts that don't involve x_k. (There will always be enough elements to witness x_k).

So every formula is equivalent to a quantifier-free formula.

2. Suppose φ is a propositional formula; write $\varphi(p_1, \cdots p_n)$ to represent the propositional variables used in φ.

Suppose $\psi_1, \cdots \psi_n$ are first order formulas. Define $\varphi(\psi_1, \cdots \psi_n)$ inductively as follows:

If $\varphi := \varphi(p_1) = p_1$ then $\varphi(\psi_1) = \psi_1$

If $\varphi := \neg\theta(p_1, \cdots p_n)$ then $\varphi(\psi_1, \cdots \psi_n) = \neg\theta(\psi_1 \cdots \psi_n)$

If $\varphi := (\theta \,\square\, \chi)$ for some binary connective $\square$ and formulas θ, χ then

$\varphi(\psi_1, \cdots \psi_n) := (\theta(\psi_1 \cdots \psi_n) \,\square\, \chi(\psi_1 \cdots \psi_n))$.

Now if $\varphi = \varphi(p_1, \cdots p_n)$ is a tautology i.e. $\models \varphi$ then by the completeness theorem, $\vdash \varphi$ i.e. there is a formula proof of φ from the propositional axioms and rules.

Suppose that proof was $\varphi_1, \varphi_2, \cdots \varphi_n = \varphi$.

Claim: For first order formulas $\psi_1 \cdots \psi_n$,

$\varphi_1(\psi_1, \cdots \psi_n), \cdots\cdots \varphi_n(\psi_1, \cdots \psi_n)$ is a proof in the first order proof system.

Pf/ If any of the φ_i's are instances of axioms that $\varphi_i(\psi_1 \cdots \psi_n)$ is a substitution instance of that axiom which is a first order axiom. If $\varphi_k := (\varphi_j \to \varphi_i)$ for $k, j < i$ then $\varphi_j(\psi_1 \dots \psi_n)$, $(\varphi_j(\psi_1 \dots \psi_n) \to \varphi_i(\psi_1 - \psi_n))$ are by induction already legitimate lines in the proof so by Modus Ponens in first order logic, so is

$\varphi_i(\psi_1 \dots \psi_n)$

and so $\vdash \varphi(\psi_1 \dots \psi_n)$ in the first order proof system.

3. Suppose $\mathcal{M}, \mathcal{N}$ are two countable structures with equivalence relations, each with infinitely many classes and each class is infinite.

Since $\mathcal{M}$ is ctble there are ctbly many equivalence classes A_0, A_1, A_2 — and each A_i is ctbly infinite. Similarly for $\mathcal{N}$, there are equivalence classes $B_0, B_1, \dots$ and each B_i is ~~not~~ ctbly infinite.

Now pick any bijection $f_i : A_i \to B_i$ and let

$f : M \to N$ be defined by $f(a) = f_i(a)$ if $a \in A_i$.

f is a bijection since each f_i is and the A_i's and B_i's partition M and N respectively. If $a E^{\mathcal{M}} b$ then $a, b \in A_i$ for some i so $f(a), f(b) \in B_i$ and so $f(a) E^{\mathcal{N}} f(b)$. If ~~a,b~~ ~~a,b~~ $a E^{\mathcal{N}} b$ then $a, b \in B_i$ for some i so $f^{-1}(a), f^{-1}(b) \in A_i$ so $f^{-1}(a) E^{\mathcal{M}} f^{-1}(b)$.

So f is an isomorphism.

Now suppose $\mathcal{M}, \mathcal{N}$ are two models of the theory of an equivalence relation with inf. classes all inf. There are $\mathcal{M}_0, \mathcal{N}_0$ ctble so that $\mathcal{M} \equiv \mathcal{M}_0$ and $\mathcal{N} \equiv \mathcal{N}_0$. But we just showed that $\mathcal{M}_0 \cong \mathcal{N}_0$ so

$$\mathcal{M} \equiv \mathcal{M}_0 \cong \mathcal{N}_0 \equiv \mathcal{N} \quad \text{so} \quad \mathcal{M} \equiv \mathcal{N}.$$

4. Add a constant symbol c to the language of $\mathbb{R}$ and consider the set of sentences

$$\Sigma = Th(\mathbb{R}) \cup \{ n < c : n \in \mathbb{N} \} \quad \text{where } n = \underbrace{1 + \cdots + 1}_{n \text{ times}}$$

If $\Sigma_0 \subseteq \Sigma$ is finite then there is N s.t.

$\Sigma_0 \subseteq Th(\mathbb{R}) \cup \{ n < c : n \leq N \}$ and so $(\mathbb{R}, N+1)$ satisfies Σ_0. By compactness, then Σ is satisfiable by some ctble model $\mathbb{R}^*$.

This $\mathbb{R}^*$ in the language of $\mathbb{R}$ satisfies $Th(\mathbb{R})$ and is not contained in $\mathbb{R}$ since the realization of the constant is greater than n for all $n \in \mathbb{N}$.

($\mathbb{R}^*$ is a ctble real ordered field which is not Archimedean)