Assignment 1, Math 701
Due Sept. 25, in class

1. Here is a presentation of dimension which generalizes the notion of dimension from vector spaces and we will use later in the course.

   Given a set X, we call a map $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ a pre-geometry or a matroid on X if it satisfies the following:

   (a) (Non-triviality) If $A \subseteq X$ then $A \subseteq \mathrm{cl}(A)$.

   (b) (Monotonicity) If $A \subseteq B \subseteq X$ then $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$.

   (c) (Idempotent) $\mathrm{cl}(A) = \mathrm{cl}(\mathrm{cl}(A))$ for all A.

   (d) (Finite Character) If $a \in \mathrm{cl}(A) \subseteq X$ then $a \in \mathrm{cl}(A_0)$ for some finite $A_0 \subseteq A$.

   (e) (Exchange) If $a \in \mathrm{cl}(B \cup \{b\}) \setminus \mathrm{cl}(B)$ then $b \in \mathrm{cl}(B \cup \{a\}) \setminus \mathrm{cl}(B)$.

   Given a pre-geometry cl on X, we say that a set $A \subseteq X$ is independent if for all $a \in A$, $a \notin \mathrm{cl}(A \setminus \{a\})$. Show that if (X, cl) is a pre-geometry and $A = \mathrm{cl}(B)$ for some finite set B then the size of a maximal independent subset of A is finite and does not depend on the choice of the maximal independent set.

2. We know that for finite-dimensional vector spaces all bases have the same size. Let's show this for all vector spaces. Suppose that V is an F-vector space and both X and Y are infinite bases for V. Show that for every $x \in X$, there is a finite $Y_0 \subset Y$ such that $x \in \langle Y_0 \rangle$ and for any such Y_0 there are only finitely many $x \in X$ in the span of Y_0. Conclude that $|X| = |Y|$ (the cardinality of X is the same as the cardinality of Y).

3. Suppose that V is an F-vector space. Consider V^*, the dual space, containing all linear transformations $f : V \to F$ with the natural addition and scalar multiplication. Show that V^* is an F-vector space. Define a canonical embedding of V into V^{**} and show that if V is finite dimensional then V is isomorphic to V^*.

4. Here is a definition of equivalence of categories I didn't give in class (it avoids the need for saying "natural transformation"): We say that two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if there is a functor $F : \mathcal{C} \to \mathcal{D}$ such that

(a) for every x, y objects of $\mathcal{C}$, F induces a bijection between $\hom_{\mathcal{C}}(x, y)$ and $\hom_{\mathcal{D}}(F(x), F(y))$, and

(b) for every object y in $\mathcal{D}$ there is an x in $\mathcal{C}$ such that y is isomorphic to $F(x)$.

Fix a field F. Suppose that $\mathcal{D}$ is the category with the natural numbers, N, as objects and for $m, n \in N$, $\hom(m, n)$ is the set of $n \times m$ matrices over F. Let $\mathcal{C}$ be the category of finite dimensional F vector spaces with linear transformations as morphisms. Prove that $\mathcal{C}$ and $\mathcal{D}$ are equivalent categories.

5. (Pullbacks) Show that whenever M, N and P are R-modules and $f : N \to M$, $g : P \to M$ are homomorphisms then there is an R-module S and homomorphisms $j : S \to N$ and $k : S \to P$ such that $fj = gk$ and moreover whenever S' and homomorphisms $j' : S' \to N$ and $k' : S' \to P$ satisfy $fj' = gk'$ there is a unique $h : S' \to S$ such that $j' = jh$ and $k' = kh$.