Assignment 2, Math 701
Due Oct. 21, in class

1. We saw in class that for a unital ring R, the free object on a set of generators X is isomorphic to $\coprod_X R$. I mentioned that $\prod_X R$ is not always free. Sometimes it is. For instance, for a field F, the F-module $\prod_X F$ is an F-vector space so has some basis and is hence, free. Let's show in the case of abelian groups products of free objects are not necessarily free. Specifically, we will show that $\prod_N Z$ is not free on any set of generators. We do this in a number of steps. One thing that you will need to know for this exercise is that $\prod_N Z$ is uncountable and that any countably generated abelian group is countable.

   (a) Show that any free abelian group is torsion-free (if $nx = 0$ then either $n = 0$ or $x = 0$) and it contains no non-zero divisible elements (x is said to be divisible if for every n, there is some y such that $ny = x$).

   (b) Suppose that A is a free abelian group on the generators $X, Y \subseteq X$ and B is the subgroup generated by Y. Show that A/B is free on the image of the generators from $X \setminus Y$.

   (c) Suppose X is a set of generators for $\prod_N Z$. Show that there is a countable set $Y \subseteq X$ such that $\coprod_N Z \subseteq B$ where B is the subgroup generated by Y.

   (d) Consider the set S of elements of $\prod_N Z$, $(a_k : k \in N)$ with the property that $k!$ divides a_k. Show that S is uncountable. (Hint: S is bijective with $\prod_N Z$).

   (e) Show that if $\coprod_N Z \subseteq B \subseteq \prod_N Z$ where B is countable then $\prod_N Z/B$ contains a divisible element by choosing something in S but not in B and showing it is divisible in the quotient.

   (f) Conclude that $\prod_N Z$ is not free on any set of generators.

2. Show that if R is a commutative ring with 1 and I and J are ideals of R then $R/I \otimes_R R/J \cong R/(I + J)$.

3. Remember the R-algebras? Let's see that tensor products answer the question we asked about R-algebras. Fix R a commutative ring with 1. Consider two R-algebras A and B. By the definition of R-algebra,

A and B can be thought of as R-modules so form $A \otimes_R B$. We would like to define an R-algebra structure on this module.

(a) Show that one can define a multiplication by linearity and declaring that $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$.

(b) Show that this R-algebra has the co-product like property as follows: Suppose that C is an R-algebra and $f : A \to C$ and $g : B \to C$ are R-algebra homomorphism whose ranges commute in C. Then there is a unique R-algebra homomorphism $h : A \otimes_R B \to C$ such that $f = h \cdot i_A$ and $g = h \cdot i_B$ where $i_A : A \to A \otimes_R B$ sending a to $a \otimes 1$ and $i_B : B \to A \otimes_R B$ sending b to $1 \otimes b$.

4. Show that if $R \subseteq S$, both commutative rings with 1 then as R-algebras, $S[x] \cong R[x] \otimes_R S$.

5. We saw that tensor products commute with coproducts. For R-modules, finite products are the same as finite coproducts so tensor products commute with finite products as well. Let's see that they don't commute with arbitrary products. Specifically prove that as abelian groups

$$\prod_{n \in N} (Q \otimes Z/p_n Z) \ncong Q \otimes \prod_{n \in N} Z/p_n Z$$

where p_n is the list of primes in Z.