Assignment 1, Math 4LT3

Due Friday, Jan. 31, by email (please send actual scans or pdfs; no photos)

1. As we discussed in class, the standard manner of capturing graphs in model theory as certain types of binary relations doesn't work if you have multiple edges. I sketched a suggestion of how this could be done. Write this up and include sentences that such a structure must satisfy in order the it captures the class of graphs with multiple edges (and possibly loops).

2. We will show that every field has an algebraically closed extension (which is actually algebraic over the original field).

   (a) Start with any field F and then form the ring $R = F[X_f : f \in Irr(F)]$ where $Irr(F)$ is the set of irreducible polynomials over F. Let I be the ideal generated by $\{f(X_f) : f \in Irr(F)\}$. Show that I is a proper ideal of R.

   (b) Choose $M \supset I$ a maximal ideal in R and show that R/M can be thought of as an algebraic field extension of F in which every irreducible polynomial over F has a solution.

   (c) Now form a chain $F = F_0 \subset F_1 \subset F_2 \ldots F_n \subset \ldots$ such that F_{n+1} is an algebraic extension of F_n which contains a solution for every irreducible polynomial over F_n. Conclude that $\bigcup F_n$ is an algebraically closed field which is algebraic over F.

   (d) It turns out that if $F \subset K$ are fields such that K is algebraically closed and algebraic over K then K is unique up to isomorphism over F and is called the algebraic closure of F.

3. We wrote out sentences in the language of fields which were satisfied by an algebraically closed field of characteristic p where p is some prime or 0. We want to show that this set of sentences is complete.

   (a) We need to define what is called a transcendental set of elements inside a field: $X \subset F$ is called transcendental if for all $x \in X$, x is not algebraic over $X \setminus \{x\}$. Show that if $X \subset F$ is a maximal transcendental set the F is algebraic over X.

   (b) Suppose that $X \subset F$ and $Y \subset G$ are maximal transcendental sets and F and G are algebraically closed of characteristic p. Show that if $f : X \to Y$ is a bijection then f can be extended to an isomorphism from F to G.

(c) Now show that if F and G are algebraically closed fields of characteristic p then there are F' and G' of the same uncountable cardinality such that $F \prec F'$ and $G \prec G'$. Conclude that $F' \cong G'$ and so F and G have the same theory.

4. Prove the Łoś theorem

5. Ultraproducts are used in other areas of mathematics; here is a fairly typical example.

   (a) Suppose that $r_i \in R$ for all $i \in I$ is a bounded family of real numbers. Fix an ultrafilter U on I. Prove that there is a unique number r such that for every $\epsilon > 0$, $\{i \in I : |r - r_i| < \epsilon\} \in U$. We call r the ultralimit of the sequence r_i along the ultrafilter U and write $r = \lim_{i \to U} r_i$.

   (b) Suppose that (X_i, d_i) is a metric space for each $i \in I$ and fix a point $a_i \in X_i$. Consider the set X be the set of all I-indexed sequences $\{\langle x_i : i \in I\rangle$ such that $x_i \in X_i$ for all $i \in I$ and there is some M such that $d_i(x_i, a_i) \leq M$ for all $i \in I$. Fix an ultrafilter U on I and define d on X by

$$d(\bar{x}, \bar{y}) = \lim_{i \to U} d_i(x_i, y_i)$$

   Show that d is a pseudo-metric on X; the quotient of X by this pseudo-metric is the metric ultraproduct of the X_i's.