Quantifier elimination

Definition

We say that a theory T has quantifier elimination if for any formula $\varphi(\bar{x})$ and $\epsilon > 0$ there is a quantifier-free formula $\psi(\bar{x})$ such that

$$\sup_{\bar{x}} |\varphi(\bar{x}) - \psi(\bar{x})| \leqslant \epsilon$$

holds in all models of T.

Theorem

Suppose that T is a complete theory in a separable language. T has quantifier elimination iff whenever M and N are separable models of T, A is a finitely generated substructure of both M and N and U is a non-principal ultrafilter on $\mathbb{N}$ then M embeds into N^U fixing A.

Example 1: Urysohn space

- For this example we will only consider metric spaces with metrics bounded by 1.
- We say that a separable metric space X is universal if every separable metric space can be embedded into X; it is homogeneous if whenever f is a finite isometry on X, it can be extended to an automorphism.
- We constructed a separable metric space which is both universal and homogeneous.
- For every possible finite metric configuration $\bar{r} = r_{ij}$ for $1 \leqslant i, j \leqslant n$ there is a formula, $C_{\bar{r}}(\bar{x})$, the configuration formula for $\bar{r}$ written as

$$max_{i,j}|d(x_i, x_j) - r_{ij}|$$

which measures how far a tuple $\bar{x}$ is from realizing the given configuration.

Consequences for Urysohn space

- The theorem from last time was that the theory of Urysohn space was axiomatized by the sentences expressing extendability of metric configurations.
- The theory of Urysohn space has quantifier elimination.
- We say that a theory T is separably categorical if any two separable models of T are isomorphic.
- The theory of Urysohn space is separably categorical.
- Corollary: $C_{\bar{r}}(\bar{x})$ has a definable zero set.
- Exercise: Compute the distance formula to a the zero set of a given configuration.

Hilbert space reminder

- A Hilbert space H is a complete complex inner product space.
- As a metric structure we formally think of a Hilbert space in the language:
  - The family of bounded metric structures B_n for all $n \in N$ together with functions $i_{m,n} : B_m \to B_n$ for $m < n$;
  - the family of functions 0, λ_n for $\lambda \in C$ and $n \in N$ and $+_{m,n}$ and $-_{m,n}$ for all $m, n \in N$; and
  - the family of relations $re(\langle -, - \rangle)_{m,n}$ and $im(\langle -, - \rangle)_{m,n}$ for $m, n \in N$ along with a metric symbol for each sort.
- For a given Hilbert space H, the standard interpretation of these symbols is B_n is the ball of radius n centred at 0; $i_{m,n}$ are the inclusion maps from B_m to B_n and all the functions and relations are interpreted as their restrictions to the corresponding balls.

Axioms for Hilbert space

- There is a large number of axioms expressing the fact that we are dealing with a complex inner product space; these axioms are all universal (the have only sup quantifiers).
- For instance, we have $\sup_{x \in B_1} \sup_{y \in B_1} d_{B_2}(x +_{1,1} y, y +_{1,1} x)$ evaluates to 0 and partially expresses that + is commutative.
- We also have relationships between the inner product and the metric:

$$\sup_{x \in B_n} \sup_{y \in B_n} (d_{B_n}(x, y)^2 - re(\langle x - y, x - y \rangle))$$

- We also have $\sup_{x \in B_1} (d(x, 0) \dot{-} 1)$.
- Are these all the axioms that we need? No.

Axioms for Hilbert space, cont'd

- The problem is that we need to know that the image of B_1 in B_n is exactly those things in B_n with distance 1 from 0.
- The needed axioms look like this: for $n \in N$, $r \leqslant n$ and m is the least integer greater than n/r

$$\sup_{x \in B_n} \min\{r \dot{-} d_{B_n}(x, i_{1,n}(0)), \inf_{y \in B_1} d_{B_m}(i_{1,m}(y), \frac{1}{r} i_{n,m}(x))\}$$

- We can axiomatize being infinite-dimensional (exercise).
- The theory of infinite-dimensional Hilbert space is separably categorical and has quantifier elimination.
- As a consequence, all complete quantifier-free types have definable zero sets. What are the distance functions here?

Ehrenfeucht-Fraïssé games

- How can we tell if two metric structures M and N in a language L are elementarily equivalent?
- In the discrete first order case, one could theoretically play a game to determine this; here are the details in the continuous setting:
- Fix $\varphi_1(\bar{x}), \ldots, \varphi_k(\bar{x})$ atomic formulas in the variables $x_1, \ldots, x_n$. We fix $\epsilon > 0$ and a finite ϵ-covers $\mathcal{C}_1, \ldots, \mathcal{C}_k$ of the ranges of $\varphi_1, \ldots, \varphi_k$ made up of closed intervals, with interior, of length at most ϵ. No three intervals intersect and no endpoint is an endpoint of two intervals. We will call this the data for the EF-game.
- The EF-game of length n with respect to this data is played as follows:

Ehrenfeucht-Fraïssé games, cont'd

- Player 1 chooses either $a_1 \in M$ or $b_1 \in N$ respecting the sort of x_1; player 2 chooses $b_2 \in N$ or $a_2 \in M$ respectively.
- Player 1 and Player 2 alternate in this manner until they have produced two sequences $a_1, \ldots, a_n \in M$ and $b_1, \ldots, b_n \in N$.
- Player 2 wins if for all i there is some $C \in \mathcal{C}_i$ such that $\varphi_i(\bar{a}), \varphi_i(\bar{b}) \in C$.

Theorem

$M \equiv N$ iff Player 2 has a winning strategy for all EF-games.

Proof of the Theorem

- Let's prove right to left. First of all we generalize the notion of an EF-game to be just as described only now $\varphi_1, \ldots, \varphi_k$ can be any formulas.
- Claim: Assuming one has a winning strategy for all possible data for the atomic game then one has a winning strategy for all versions of the general game.
- This will be particularly interesting when $k = 1$ and $n = 0$; in this case, we are dealing with a sentence φ.
- Since we can win the game (in no steps!), this means that φ^M and φ^N lie in the same ϵ-neighbourhoods for all ϵ i.e. $\varphi^M = \varphi^N$. So $M \equiv N$.

Proof of the Theorem, cont'd

- We prove the claim by induction on formulas; really, by induction on the complexity of the most complicated formula among $\varphi_1, \ldots, \varphi_k$. There are two cases and for simplicity we assume that $k = 1$ and $\varphi = \varphi_1$.
- The first case is that $\varphi = f(\psi_1, \ldots, \psi_l)$ for some continuous function f.
- In this case, choose δ corresponding to ϵ from the uniform continuity modulus of f on the ranges of $\psi_1, \ldots, \psi_l$. Now fix finite δ-covers of these ranges $\mathcal{D}_1, \ldots, \mathcal{D}_l$ so that for $D_1 \in \mathcal{D}_1, \ldots, D_l \in \mathcal{D}_l$, $f(D_1 \times \ldots \times D_l) \subseteq C$ for some $C \in \mathcal{C}$.

Proof of the Theorem, cont'd

- In order to win the original game, we play the winning strategy for the game corresponding to the $\psi_1, \ldots, \psi_l$, δ and the $\mathcal{D}_i$'s.
- At the end of that game, we have sequences $\bar{a} \in M$ and $\bar{b} \in N$. We can find $D_1 \in \mathcal{D}_1, \ldots, D_l \in \mathcal{D}_l$ such that $(\psi_1(\bar{a}), \ldots, \psi)_l(\bar{a})), (\psi_1(\bar{b}), \ldots, \psi_l(\bar{b})) \in D_1 \times \ldots \times D_l$. So $f(\psi_1(\bar{a}), \ldots, \psi_l(\bar{a})), f(\psi_1(\bar{b}), \ldots, \psi_l(\bar{b})) \in C$ for some $C \in \mathcal{C}$.

Proof of the Theorem, cont'd

- The second case is when $\varphi = \sup_y \psi(y, x_1, \ldots, x_n)$ (or inf but the cases will be symmetric so we will only do the sup case).
- Now what we know is that we can win the $n+1$-game with ψ replacing φ and all the same data (a word about the cover).
- Use the winning strategy for this game to play the original length n game. Why do we win?

Proof of the Theorem, cont'd

- Suppose that we have chosen $\bar{a}$ and $\bar{b}$ according to the winning strategy. We need to see that if $\sup_y \psi(y, \bar{a}), \sup_y \psi(y, \bar{b}) \in C$ for some C in our cover. Suppose $\sup_y \psi(y, \bar{a}) < \sup_y \psi(y, \bar{b})$.
- For this we enlist Player 1's help. Pick b_n such that $\psi(b_n, \bar{b})$ increases to $\sup_y \psi(y, \bar{b})$. For each n, we can find a_n and $C_n \in \mathcal{C}$ so that $\psi(a_n, \bar{a}), \psi(b_n, \bar{b}) \in C_n$. Since $\mathcal{C}$ is finite, there is a single C which contains infinitely many b_n's. It follows that $\sup_y \psi(y, \bar{b}) \in C$ and so is $\sup_y \psi(y, \bar{a})$.

Proof of the Theorem, cont'd

- For the other direction, a sketch: we show by induction on n that we can win any general EF game. The assumption is that $M \equiv N$.
- The case $n = 0$: This is the case of sentences and this follows by elementarity.
- Now suppose we are dealing with the case $n = k + 1$. For simplicity let's assume that we have only one formula φ and a covering of its range $\mathcal{C}$.
- In fact, the right way to look at $\mathcal{C}$ is as an increasing sequence of points $r_0 < r_1 < \ldots < r_t$ representing the endpoints of the intervals present in $\mathcal{C}$.
- Consider the formulas ψ_i, for each $i < t$, in k variables given by

$$\inf_y \max\{r_i \dot{-} \varphi(x_1, \ldots, x_k, y), \varphi(x_1, \ldots, x_k, y) \dot{-} r_{i+1}\}$$

Proof of the Theorem, cont'd

- We now play the k game with all these new formulas and with a δ-cover where $\delta > 0$ but less than half the minimum of $|r_{i+1} - r_i|$ for $i < t$.
- Now the strategy for the original $k+1$ game is to follow the winning strategy for the above k game for the first k turns. This will produce two sequences $\bar{a} \in M$ and $\bar{b} \in N$. Then, if Player 1 chooses $a = a_{k+1}$ we fix i such that a witnesses $\psi_i(\bar{a}) = 0$. By induction, $\psi_i(\bar{b}) \leqslant \delta$. We can therefore find a witness $b \in N$ such that

$$r_i - 2\delta \leqslant \varphi(\bar{b}, b) \leqslant r_{i+1} + 2\delta$$

- We are finished by the following picture: