Assignment 2, Math 712

Due Feb. 13, emailed to me in scanned pdf format (no pictures please)

1. Consider the class G_{nt} of all finite triangle-free graphs (nt for no triangles). Show that this is a Fraïssé class i.e. it is closed under isomorphisms, subgraphs, amalgamation, and for every n, there are, up to isomorphism, only finitely many triangle-free graphs of size n. Construct a generic countable graph H_{nt} as we did with the random graph with the property that it is universal for the class G_{nt} and is ultrahomogeneous. Show that there is only one countable graph with this property. Write out axioms for this class and conclude that these axioms are complete.

   Do some literature research and find out what you can about the almost sure theory of triangle-free graphs. Is it the same as the theory of H_{nt}? Is the theory of H_{nt} pseudo-finite? Hint: some of this is an open research question.

2. Prove the Łoś Theorem for metric spaces. That is, show that if $X = \prod_{\mathcal{U}} X_i$ where the X_i's are an I-indexed family of uniformly bounded metric spaces then whenever $\varphi(x_1, \ldots, x_n)$ is a formula in the language of metric spaces and $a^1, \ldots, a^n \in X$ then

$$\varphi^X(a^1, \ldots, a^n) = \lim_{\mathcal{U}} \varphi^{X_i}(a^1_i, \ldots, a^n_i).$$

3. Suppose that (X_i, d_i) for $i \in I$ is a uniformly bounded I-indexed family of metric spaces and f_i is a continuous function of one variable on X_i for each i (continuous with respect to d_i). Algebraically we can define $X' = \prod_I X_i$ and define f coordinate-wise on X' via the f_i's. If $\mathcal{U}$ is an ultrafilter on I, then $X = \prod_{\mathcal{U}} X_i$ is a quotient of X'. What condition do we need to put on the f_i's so that f is well-defined on this quotient?

4. Show that the Urysohn sphere, $\mathcal{U}$, is ultrahomogeneous. That is, suppose that $X \subset Y$ are both finite $[0, 1]$-metric spaces and $X \subset \mathcal{U}$. Then there is a Y', $X \subset Y' \subset \mathcal{U}$ with $Y \cong Y'$ with X fixed.

5. I made the claim in class that there is no effective difference between pseudo-compact and pseudo-finite in continuous logic. Let me try to justify that claim. Suppose that X is a compact metric space. For each

n, choose $a_1, \ldots, a_{k_n} \in X$ such that the open $\frac{1}{n}$-balls centered at the a_i's cover X; we can assume that the a_i's are at least $\frac{1}{n}$ apart (Why?). Let X_n be the subspace consisting of $\{a_i : i \leq k_n\}$. Let $\mathcal{U}$ be any non-principal ultrafilter on $\mathbf{N}$ and prove that $X \cong \prod_{\mathcal{U}} X_n$.