The language of a metric structure

A language L consists of

- a set S called sorts;
- $\mathcal{F}$, a family of function symbols. For each $f \in \mathcal{F}$ we specify the domain and range of f: $dom(f) = \prod_{i=1}^{n} s_i$ where $s_1, \ldots, s_n \in S$ and $rng(f) = s$ where $s \in S$. Moreover, we also specify a continuity modulus. That is, for each i we are given $\delta_i^f : [0, 1] \to [0, 1]$; and
- $\mathcal{R}$, a family of relation symbols. For each $R \in \mathcal{R}$ we are given the domain $dom(R) = \prod_{i=1}^{n} s_i$ where $s_1, \ldots, s_n \in S$ and the $rng(R) = K_R$ for some closed interval K_R. Moreover, for each i, we specify a continuity modulus $\delta_i^R : [0, 1] \to [0, 1]$.
- For each $s \in S$, we have one special relation symbol d_s with domain $s \times s$ and range of the form $[0, B_s]$. It's continuity moduli are the identity functions.

Ultraproducts

Suppose that

$$M_i = (\{(X^i_s, d^i_s) : s \in S\}, \{R^i : R \in \mathcal{R}\}, \{f^i : f \in \mathcal{F}\})$$

is an I-indexed family of metric structures for a language L. Fix an ultrafilter U on I. The ultraproduct $M = \prod_{i \in I} M_i/U$ is defined to be the L-structure with sorts, for $s \in S$, given by

$$\prod_{i \in I} (X^i_s, d^i_s)/U$$

and functions and relations given by

$$\lim_{i \to U} f^i \text{ and } \lim_{i \to U} R^i$$

for $f \in \mathcal{F}$ and $R \in \mathcal{R}$.

Terms

Definition

We define terms in a language L, their domains and ranges, and continuity moduli inductively:

- Any single variable x of sort s is a term. It has domain and range s and the identity function as continuity moduli.
- If f is a function symbol in L with $dom(f) = \prod_{i=1}^{n} s_i$ and $rng(f) = s$, and τ_i for $i = 1, \ldots, n$ are terms where $rng(\tau_i) = s_i$ for all i. Then $f(\tau_1, \ldots, \tau_n)$ is a term. The domain, range and uniform continuity modulus are those obtained by composition.

Formulas

Definition

We define formulas, their domains and ranges, and continuity moduli inductively:

- If R is a relation symbol in L with $dom(R) = \prod_{i=1}^{n} s_i$ and $rng(R) = K_R$, and τ_i are terms where $rng(\tau_i) = s_i$ for all i then $R(\tau_1, \ldots, \tau_n)$ is a formula. The domain, range and continuity moduli are those obtained by composition.
- If $\varphi_i(\bar{x})$ is a formula with range K_{φ_i} for all $i \leq n$ and $f : R^n \to R$ is a continuous function then $f(\varphi_1, \ldots, \varphi_n)$ is a formula with range $f(\prod_{i=1}^{n} K_{\varphi_i})$ and domain and continuity moduli determined by composition.
- If φ is a formula and x is a sorted variable then $\sup_x \varphi$ and $\inf_x \varphi$ are both formulas. The sort of x is removed from the domain; the range and continuity moduli for the remain variables stay the same.

Interpretations

Fix a metric structure M for a language L.

- Terms are interpreted by composition inductively as in the definition.
- For the formula $R(\tau_1(\bar{x}), \ldots, \tau_n(\bar{x}))$ where R is a relation in L and $\tau_1, \ldots, \tau_n$ are terms, its interpretation is given, for every appropriate $\bar{a} \in M$, by

$$R^M(\tau_1^M(\bar{a}), \ldots, \tau_n^M(\bar{a}))$$

- If $\varphi_i(\bar{x})$ is a formula for all $i \leq n$ and $f : R^n \to R$ is a continuous function then if ψ is the formula $f(\varphi_1, \ldots, \varphi_n)$ then $\psi^M = f(\varphi_1^M, \ldots, \varphi_n^M)$.

Interpretations cont'd

- Suppose $\varphi(x, \bar{y})$ is a formula and $\bar{a} \in M$ is a tuple from M appropriate for the variables $\bar{y}$ and x is of sort s. Then

$$\sup_x \varphi(x, \bar{a}) := \sup\{\varphi(b, \bar{a}) : b \in X_s\}$$

and

$$\inf_x \varphi(x, \bar{a}) := \inf\{\varphi(b, \bar{a}) : b \in X_s\}$$

Basic properties

Proposition

In an L-structure M

- *the interpretations of the terms are uniformly continuous functions with uniform continuity modulus as specified by the definition of the term;*
- *all formulas when interpreted in M, define uniformly continuous functions with domains, range and uniform continuity modulus as specified by the definition.*

A sentence is a formula with no free variables. It is a consequence of the proposition that any sentence in L takes on a value in a metric structure in a compact interval specified by L and this interval is independent of the given structure.

Łoś Theorem

Theorem

Suppose M_i are metric structures for all $i \in I$, U is an ultrafilter on I, $\varphi(\bar{x})$ is a formula and $\bar{a} \in \prod_{i \in I} M_i / U$ then

$$\varphi(\bar{a}) = \lim_{i \to U} \varphi^{M_i}(\bar{a}_i)$$

Conditions

Definition

Fix a language L.

- For a sentence φ in L, a condition is an expression of the form $\varphi \leq r$ or $\varphi \geq r$ for a real number r.
- We say that a condition $\varphi \leq r$ (resp. $\varphi \geq r$) holds in an L-structure M if $\varphi^M \leq r$ (resp. $\varphi^M \geq r$).
- For $\epsilon > 0$ and a condition $\varphi \leq r$ (resp. $\varphi \geq r$), we call the condition $\varphi \leq r + \epsilon$ (resp. $\varphi \geq r - \epsilon$) the ϵ-approximation of the condition. For a set Σ, its ϵ–approximation is the set of ϵ-approximations of all of its elements.

Satisfiability

Definition

- We say a set of conditions Σ in a language L is satisfied if there is an L-structure M such that for every condition in Σ holds in M.
- We say such a Σ is finitely satisfied if every finite subset of Σ is satisfied.
- Σ is approximately finitely satisfied if for every $\epsilon > 0$ and for every finite subset Σ_0 of Σ, the ϵ-approximation of Σ_0 is satisfiable.

Compactness

Theorem

TFAE for a set of sentences Σ in a language L

- Σ *is satisfiable.*
- Σ *is finitely satisfiable.*
- Σ *is approximately finitely satisfiable.*

Hintikka sets

Definition

We call a set of conditions Σ in a language L a Hintikka set if

- it is finitely satisfiable,
- for every real number r and every sentence φ in L at least one of $\varphi \leq r$ or $\varphi \geq r$ is in Σ, and
- for every sentence of the form $\psi = \inf_x \varphi(x)$, every $\epsilon > 0$ and every real number r, if $\psi \leq r$ is in Σ then for some constant c, $\varphi(c) \leq r + \epsilon$ is in Σ.

We call the second condition "being maximal" and the third, the Henkin condition.

Canonical structure from a Hintikka set

- Notice that the maximality condition and finite satisfiability guarantees that for an sentence φ,

$$\inf\{r : \varphi \leq r \text{ in } \Sigma\} = \sup\{r : \varphi \geq r \text{ in } \Sigma\}$$

  These are well-defined since in all models of L, the value of φ is restricted to a compact interval. Call this number φ^Σ.
- Assume for simplicity that there is only one sort. We wish to put a metric structure on the set of constants C in L.
- For two constants c and c' in C, define the distance $d(c, c')$ to be $d(c, c')^\Sigma$.
- It is easy to check that defines a pseudo-metric on C since Σ is finitely satisfiable. The underlying metric space M will be the completion of C with respect to d.
- Notice that C then is dense in M.

The rest of the structure

- For any function f in L, we need to define its values on M. Assume that f is unary. We define it on C as follows:
  - For $c \in C$, $\inf_x d(x, f(c)) = 0$ is in Σ.
  - By the Henkin property, for each n there is c_n such that $d(c_n, f(c)) \leq 1/n$ is in Σ.
  - Let $f(c)$ be defined as the class of the Cauchy sequence $\langle c_n : n \in N \rangle$.
- Now extend the definition of f to all of M by continuity and prove that the continuity modulus is what is necessary for this to be an L-structure. Here you need to use finite satisfiability and the fact that every L-structure interprets f as a function with the necessary continuity modulus.
- For any relation R in L, we again define it on C. Assume R is unary for simplicity. We let $R(c) = R(c)^{\Sigma}$. One needs to check that this extends to all of M in the proper manner.

The point

Claim

For all formulas $\varphi(\bar{x})$ and all $\bar{m} \in M$, if $\bar{c}_n$ is a sequence of constants in M which tends to $\bar{m}$ then

$$\varphi(\bar{m}) = \lim_{n\to\infty} \varphi(\bar{c}_n) = \lim_{n\to\infty} \varphi(\bar{c}_n)^{\Sigma}$$

The proof

One checks by induction on formulas that every formula φ is uniformly continuous and on C satisfies $\varphi(\bar{c}) = \varphi(\bar{c})^{\Sigma}$. To handle sup one notices that $\inf_x(-\varphi) = -\sup_x \varphi$.

The Henkin construction

- Let's use a Hintikka set to prove the compactness theorem. Suppose that you have an approximately finitely satisfiable set Σ.
- Without loss, by replacing it by its approximations, we can assume that Σ is finitely satisfiable.
- How do you get a Hintikka set?
- Augment your language with an immense number of new constants.
- Inductively (transfinitely) define a sequence of sets starting with Σ and require that at each stage α, Σ_α is finitely satisfiable. This property will clearly survive through limit stages.

The Henkin construction, cont'd

- Enumerate all sentences in the new language and all real numbers and make sure that for each φ and each r, at some stage, either $\varphi \leq r$ or $\varphi \geq r$ is added. This is possible by the finite satisfaction condition. If one achieves this, we get maximality.
- Finally, to take care of the Henkin condition, enumerate all possible sentences of the form $\inf_x \varphi(x)$ and all real numbers r. Make sure that if $\inf_x \varphi(x) \leq r$ gets into Σ_α at some stage, at some future stage $\beta + 1$, we add $\varphi(c)$ to Σ_β for some c that has not been mentioned in Σ_β.
- We get a Hintikka set whose canonical structure satisfies the original Σ.