The Main Theorem

Theorem

- *T is a complete continuous theory in L;*
- *T is contained in T', a complete continuous theory in L' containing L;*
- *the forgetful functor from $Mod(T')$ to $Mod(T)$ is an equivalence of categories, then*
- *every sort in L' is in definable bijection with a definable zero set in L.*

This will tell us by stable embeddedness that every L' function and relation can also be expressed as a definable predicate in L.

A sketch of the proof

- Fix a saturated model M of T' and suppose $c \in S(M)$, S a sort from L'. Consider $\varphi(\bar{x}, c)$ where $\bar{x}$ ranges over sorts from L.
- By stable embeddedness and compactness, for each n, there are $\psi_i(\bar{x}, \bar{y}_i)$ for $i = 1, \ldots, m_n$ such that

$$\min_i \inf_{\bar{y}_i} |\varphi(\bar{x}, c) - \psi_i(\bar{x}, \bar{y}_i)| \leqslant \frac{1}{2^n}$$

- Let $\bar{\psi}_n$ be the single formula which codes the canonical parameters for $\psi_1 \ldots \psi_{m_n}$ and $S_{\bar{\psi}_n}$ be the sort of those canoncial parameters.

$$\bar{S}_\varphi = \prod_n S_{\bar{\psi}_n}$$

- The definable predicate $\varphi(\bar{x}, c)$ is captured by an element of $\bar{S}$, a sort entirely in T^{eq}.

Not quite

- The identification of $\varphi(\bar{x}, c)$ with an element of $\bar{S}_\varphi$ may not be canonical; we fix this with "forced convergence".
- For a sequence of real numbers a_n for $n \in \mathbb{N}$ we define numbers b_n such that $b_{n+1} = a_{n+1}$ if $b_n - 2^{-n} \leqslant a_n \leqslant b_n + 2^{-n}$. If $a_n \geqslant b_n + 2^{-n}$ then let $b_{n+1} = b_n + 2^{-n}$ and if $a_n \leqslant b_n - 2^{-n}$ then let $b_{n+1} = b_n - 2^{-n}$.
- This produces a continuous function from sequences of real numbers to fast converging Cauchy sequences; we identify sequences which converge to this same forced limit.
- This gives a formula on $\bar{S}_\varphi$, $\Psi(\bar{x}, \bar{c})$ which outputs the the same forced limit when we compute the limit of the sequence of ψ_n's.
- $\bar{S}_\varphi$ will be the sort in which we quotient by the canonical parameters for Ψ.

A sketch of the proof, cont'd

- Consider

$$\Sigma_n = \{\sup_{\bar{x}} |\varphi(\bar{x}, c) - \varphi(\bar{x}, c')| \leqslant \frac{1}{k} : k \in \omega, \bar{x} \in L\} \cup \{d_S(c, c') \geqslant \frac{1}{n}\}$$

- Σ_n is inconsistent by assumption for every n so by compactness there are countably many formulas $\varphi_i(\bar{x}, y)$ such that if two elements of S agree on all these formulas then they are equal.
- So there is a definable injection from S into $\prod_i \bar{S}_{\varphi_i}$ and we can identify S with the definable zero set which is the range of this map.

4 ways to say stable: definition 1

Definition

We say that a complete theory T is λ-stable if whenever $M \models T$, $\chi(M) \leqslant \lambda$ then $\chi(S(M)) \leqslant \lambda$ where the type space has the metric topology.
T is stable if it is λ-stable for some λ.

- The theory of infinite-dimensional Hilbert space is stable; in fact it is $\aleph_0$-stable.
- If M is the infinite dimensional separable Hilbert space then $S(M)$ or more precisely the space of 1-types in x over the unit ball of M is determined, by quantifier elimination, essentially by specifying the inner product of x with each element of an orthonormal basis for M.
- There are clearly $2^{\aleph_0}$ many types but what is the density of these types?

Definition 1, cont'd

- Suppose that $p(x)$ is any type over M. $p(x)$ determines the orthogonal projection of x onto M; call this u_p. Otherwise, p determines the length of $x - u_p$ which is an element orthogonal to M.
- Since M is separable, we can specify countably many types with u_p from a countable dense set and we can have $|x - u_p|$ be rational. This set of types is dense in $S(M)$.
- To use this definition of stability, one needs to know a lot of information about the types which is usually only available if you have some form of quantifier simplification.

Definition 2

Definition

Suppose that T is a complete theory and $\varphi(\bar{x}, \bar{y})$ is a formula. T is said to have the order property with respect to φ if there are numbers $r < s$, $M \models T$ and a sequence $\langle a_n b_n : n \in \mathbb{N} \rangle \subseteq M$ such that

$$\varphi(a_m, b_n) \leqslant r \text{ if } m \leqslant n \text{ and } \varphi(a_m, b_n) \geqslant s \text{ if } m > n$$

T is said to have the order property if it has the order property with respect to some formula.

- Urysohn space has the order property: picture.

Definition 3

- Fix a saturated model M and suppose we have a ternary relation $\downarrow$ between small subsets of M (of size $< \chi(M)$). We define a series of properties such a relation might have:
- (Invariance) For any $\sigma \in Aut(M)$, $A \downarrow_C B$ iff $\sigma(A) \downarrow_{\sigma(C)} \sigma(B)$.
- (Symmetry) $A \downarrow_C B$ iff $B \downarrow_C A$.
- (Transitivity) If $C \subseteq D$ then $A \downarrow_B C$ and $A \downarrow_{BC} D$ iff $A \downarrow_B D$.
- (Extension) If $B \subseteq C \subseteq D$ and $A \downarrow_B C$ then there is $\sigma \in Aut(M/C)$ such that $\sigma(A) \downarrow_C D$.

Definition 3, cont'd

- (Finite character) $A \downarrow_C B$ iff for all finite $A_0 \subseteq A$, $B_0 \subseteq B$, $A_0 \downarrow_C B_0$.
- (Local character) There is a κ such that for all A and B, there is $B_0 \subseteq B$ such that $\chi(B_0) \leqslant \kappa + \chi(A)$ and $A \downarrow_{B_0} B$.
- (Stationarity) For any A and $N \prec M$, if $N \subseteq C$ and $\sigma \in Aut(M/N)$ such that $A \downarrow_N C$ and $\sigma(A) \downarrow_N C$ then there is $\mu \in Aut(M/C)$ such that $\mu(A) = \sigma(A)$.

Definition

We say that M or $Th(M)$ supports a stationary independence relation if it has a ternary relation between small subsets which satisfies all of the above conditions.

Definition 3, cont'd

- This notion is a weakening of the van der Waerden axioms for a dependence relation. You can't define dimension using this relation; you do have the exchange property.
- The theory of an infinite dimensional Hilbert space supports a stationary independence relation. Define $\downarrow$ by $A \downarrow_C B$ iff if $a \in \langle AC \rangle$ and a is orthogonal to C then a is orthogonal to B.
- Notice that if T supports a stationary independence relation then T is stable:
- Fix λ such that $\lambda^\kappa = \lambda$ where $\kappa \geqslant \chi(L)$ and κ satisfies the local character axiom. Choose $M \models T$ with $\chi(M) \leqslant \lambda$. If $p \in S(M)$ then p is the unique extension of some type $p\restriction_{M_0}$, $M_0 \prec M$ and $\chi(M_0) \leqslant \kappa$.
- There are at most λ^κ many possible M_0's and 2^κ many possible types over each M_0 so $|S(M)| \leqslant \lambda^\kappa = \lambda$.
- We only used some of the axioms here: invariance, stationarity, local character, transitivity.

Definition 4

- Suppose that L is separable and M is a separable L-structure. Fix non-principal ultrafilters U and V on $\mathbb{N}$ and ask if $M^U \cong M^V$.
- Unfortunately, this question is dependent a little on cardinalities. Remember that M^U is $\aleph_1$-saturated and so if $2^{\aleph_0} = \aleph_1$, it would be saturated. So then $M^U \cong M^V$ since they are elementarily equivalent.
- What if this happens even if CH does not hold? If it happens that $M^U \cong M^V$ for all non-principal ultrafilters U, V on $\mathbb{N}$ no matter what the value of the continuum, we say that these ultrapowers are necessarily isomorphic.

Main Theorem

Theorem

The following are equivalent:

1. T *is stable.*
2. T *does not have the order property.*
3. T *supports a stationary independence relation.*
4. *(L separable) For all (any) separable models of* T*, the ultrapowers with respect to non-principal ultrafilters on* $\mathbb{N}$ *are necessarily isomorphic.*

- We have seen that 3 implies 1. The rest are difficult and require the introduction of several new techniques.

Indiscernibles

Definition

Suppose that $(I, <)$ is a linear order and $\langle \bar{a}_i : i \in I \rangle$ is an I-indexed sequence in some model M. Then this sequence is said to be indiscernible if whenever $i_1 < i_2 < \ldots < i_n$ and $j_1 < j_2 < \ldots < j_n$ then $t(a_{i_1} \ldots a_{i_n}) = t(a_{j_1} \ldots a_{j_n})$.

Theorem

Suppose that M is a non-compact metric structure. Then for any $(I, <)$ there is an $M' \models Th(M)$ and an I-indexed non-constant indiscernible sequence in M'.

- Since M is not compact there is an $\epsilon > 0$ such that M is not covered by finitely many ϵ-balls. Fix an infinite set $\{a_i : i \in \mathbb{N}\}$ such that for $i \neq j$, $d(a_i, a_j) \geq \epsilon$.

Indiscernibles, cont'd

- We need to show that $Th(M)$ is satisfiable with the set of formulas, for each $\varphi(x_1, \ldots, x_n)$, $k \in \mathbb{N}$ and $i_1 < \ldots < i_n, j_1 < \ldots < j_n$ in I,

$$|\varphi(c_{i_1}, \ldots, c_{i_n}) - \varphi(c_{j_1}, \ldots, c_{i_n})| \leqslant 1/k$$

  and for every $i \neq j$ in I, $d(c_i, c_j) \geqslant \epsilon$.

- We do this by compactness so fix finitely many formulas $\varphi_1, \ldots, \varphi_m$ and $k_1, \ldots, k_m \in \mathbb{N}$. We may assume that all the formulas have the same number of free variables say $x_1, \ldots, x_n$.
- Fix finite $1/k_i$-partitions P_i of the range of φ_i. We define a colouring of n-element subsets of $\mathbb{N}$ by $P_1 \times \ldots \times P_n$: if $i_1 < \ldots < i_n$ in $\mathbb{N}$, let

$$f(\{i_1, \ldots, i_n\}) = (p_1, \ldots, p_n) \text{ iff for all } i \leqslant m, \varphi_i^M(a_{i_1}, \ldots, a_{i_n}) \in p_i$$

Indiscernibles, cont'd

- By Ramsey's Theorem, we can find a homogeneous subset of $\mathbb{N}$ such that f takes a constant value on n-element subsets of this set.
- Moreover, all the elements of the homogeneous subset are at least ϵ apart.
- We conclude that our set of formulas is satisfiable and we find M' which contains an I-indiscernible sequence.
- Example: In an infinite dimensional Hilbert space, an orthonormal set is indiscernible ordered any way you like.
- The sequence which witnessed the order property for Urysohn space was also indiscernible ordered in the way it was given.

Order implies unstable

Corollary (to the previous proof)

If T has the order property then T is unstable.

- Proof sketch: Fix $\varphi(\bar{x}, \bar{y})$ and $r < s$ which witnesses the order property. Using the same style of proof from the previous theorem we can prove that for any ordered set $(I, <)$, we can find $M \models T$ and I-indexed indiscernible sequence $\langle a_i b_i : i \in I \rangle$ such that $\varphi(\bar{a}_i, \bar{b}_j) \leqslant r$ if $i \leqslant j$ and $\varphi(a_i, b_j) \geqslant s$ if $i > j$.
- Now fix a cardinal λ and choose κ least such that $2^\kappa > \lambda$. Then $\kappa \leqslant \lambda$ and $2^{<\kappa} \leqslant \lambda$.
- Order 2^κ by $\eta < \mu$ if, for the greatest α such that $\eta\restriction_\alpha = \mu\restriction_\alpha$, $\eta(\alpha) < \mu(\alpha)$.
- Identify $2^{<\kappa}$ with those elements of 2^κ which are eventually 0.

Proof cont'd

- Pick a model M and a 2^κ-indexed indiscernible sequence $\langle a_\eta b_\eta : \eta \in 2^\kappa \rangle$ ordered by φ.
- Let $B = \{b_\eta : \eta \in 2^{<\kappa}\}$, a set of size $\leqslant \lambda$.
- For $\eta \in 2^\kappa \backslash 2^{<\kappa}$, consider all the types $t(a_\eta/B)$. Now if $\eta < \mu$, choose $\bar{\mu} \in 2^{<\kappa}$ such that $\eta < \bar{\mu} < \mu$. Then

$$\varphi(a_\eta, b_{\bar{\mu}}) \leqslant r \text{ and } \varphi(a_\mu, b_{\bar{\mu}}) \geqslant s$$

- So $t(a_\eta/B)$ and $t(a_\mu/B)$ are not equal. Moreover, if $\epsilon = \frac{s-r}{2}$ and we choose δ from the continuity modulus for φ in the $\bar{x}$-variable, we see that $t(a_\eta/B)$ and $t(a_\mu/B)$ are at least δ apart so $\chi(S(B)) \geqslant 2^\kappa > \lambda$ and $\chi(B) \leqslant \lambda$.
- B isn't a model but we could extend B to a model of the same density character and we would still have too many separated types over this model.
- So the order property implies that T is not λ-stable for any λ.

Unstable implies order

Definition

We say $p(x) \in S(M)$ is finitely determined if for every formula $\varphi(x, y)$ and every $\epsilon > 0$ there is a finite $B \subseteq M$ and $\delta > 0$ such that for all $c_1, c_2 \in M$, if

$$\max_{b \in B} |\varphi(b, c_1) - \varphi(b, c_2)| < \delta$$

then

$$|p^{\varphi(x,c_1)} - p^{\varphi(x,c_2)}| \leqslant \epsilon$$

Theorem

The following are equivalent:

1. *T is stable.*
2. *T does not have the order property.*
3. *For every $M \models T$, every type in $S(M)$ is finitely determined.*

Unstable implies order: proof

- We just proved that 1 implies 2. Let's show that 3 implies 1.
- Fix λ such that $\lambda^{\chi(L)} = \lambda$. Then if $M \models T$ and $\chi(M) \leqslant \lambda$, there are at most $\lambda^{\chi(L)} = \lambda$ many types in $S(M)$ by finite determinancy. So T is λ-stable.
- We show now that the failure of 3 implies the existence of order. So fix a type $p(x) \in S(M)$ which is not finitely determined say witnessed by a formula $\varphi(x, y)$ and $\epsilon > 0$.
- We use p to construct a sequence $a_i b_i c_i$ in M inductively; assume we have constructed these for all $i < j$.
- By assumption, we know that we can find b_j and c_j so that

$$\max_{i<j} |\varphi(a_i, b_j) - \varphi(a_i, b_j)| < \frac{\epsilon}{6} \text{ and } |p^{\varphi(x,b_j)} - p^{\varphi(x,c_j)}| > \epsilon$$

Unstable implies order: proof, cont'd

- Now by the approximate finite satisfiability of p, we can find $a_j \in M$ so that

$$|\varphi(a_j, b_j) - p^{\varphi(x,b_j)}| \leqslant \frac{\epsilon}{3} \text{ and } |\varphi(a_j, c_j) - p^{\varphi(x,c_j)}| \leqslant \frac{\epsilon}{3}$$

- So we have that if $i \leqslant j$

$$|\varphi(a_i, b_j) - \varphi(a_i, c_j)| \leqslant \frac{\epsilon}{6}$$

and for $i > j$,

$$|\varphi(a_i, b_j) - \varphi(a_i, c_j)| \geqslant \frac{\epsilon}{3}$$

- If we let $\theta(x_1 y_1 z_1, x_2 y_2 z_2) := |\varphi(x_1, y_2) - \varphi(x_1, z_2)|$ then θ orders the sequence $\langle a_i b_i c_i : i \in \mathbb{N}\rangle$.