

For this question with minimizing the area of a convex polygon on the hyperbolas, it seems like a quick way to see that the diagonals go through the origin without having to consider a bunch of cases is the following. I've attached a picture (of a non-minimal area polygon) with some colored lines that I'll refer to.

Suppose we have ABCD minimizing area, with the points labeled A, B, C, D in the order of the quadrants.

If we want to show that AC (blue) goes through the origin, take the other diagonal BD (red) and rotate it 180 degrees about the origin, to get a parallel line (orange). So, the orange line is the image under 180 degree rotation of BD, and note also that the image of A's component under this rotation is C's component, and that we know A and C must be closest to BD among the points on their respective components (otherwise they do not make the smallest triangles ABD and CBD).

Since C must be closest to BD and the orange line is parallel to BD, C must be closest to the orange line. (Side note: we know the orange line and BD don't intersect A or C's components.)

But since A is closest to BD on A's component, the image of A (which is on C's component) is closest to the orange line as well. So, C is the image of A under rotation by 180 degrees.

This happens similarly for B and D, so we know both diagonals go through the centre.