Comments on Problem Sessions 1, 2 and problems for Problem Session 3

Problem Session 1: The only problem we did in full was the following:

Suppose you choose two natural numbers at random; what is the probability that their greatest common divisor is 1? Slightly more precisely, consider all natural numbers less than N and assume a uniform distribution on all pairs (a, b) with $a, b < N$. Consider the probability that the $\gcd(a, b) = 1$ and determine the limit as N tends to infinity.

Solution If the probability of the gcd being 1 is ρ then the probability of the gcd being N is ρ/N^2; to see this, notice that the density of numbers divisible by N among all numbers is $1/N$. So the probability of choosing any pair of numbers is 1 and this equals the sum of the probabilities that that pair has any given gcd so

$$1 = \sum_{N=1}^{\infty} \rho/N^2$$

One remembers (**Putnam moment**) that

$$\sum_{N=1}^{\infty} 1/N^2 = \pi^2/6$$

and so $\rho = 6/\pi^2$.

Problem Session 2: The first problem was from the first problem session:

1. Construct a function $f : R \to R$ such that for every $x > 0$, $\lim_{n\to\infty} f(nx) = 0$ but $\lim_{x\to\infty} f(x)$ does not exist.
2. Show that if you assume above that f is continuous then $\lim_{x\to\infty} f(x) = 0$.

Proof Consider R as a rational vector space; since R is uncountable (**Putnam moment**) we can find infinitely many $x_i \in R$ for $i \in N$ which are linearly independent and such that $x_N > N$ for all N. Define f to be zero everywhere except at any x_N where the function is defined to be 1. It is clear that f does not have a limit as x tends to infinity. Let's check that for any $x > 0$, $f(nx)$ tends to zero as n tends to infinity. In fact, for any given x, there is at most one N such that for some m, $mx = x_N$. To see this, suppose not and suppose that $mx = x_M$ and $nx = x_N$. Then $x_M = nx_N/N$ which would imply that x_N and x_M are linearly independent. So $f(nx)$ is 1 at most once for any x so the limit as n tends to infinity to 0.

Now suppose that f is continuous and that $\lim_{x\to\infty} f(x)$ does not exist. This means that there is an $\epsilon > 0$ and intervals (a_i, b_i) such that $a_i > i$ and on the interval (a_i, b_i), $|f(x)|$ is greater than ϵ. Define a sequence of decreasing intervals I_i as follows: Let $I_0 = (a_0, b_0)$. If you have defined $I_n = (c_n, d_n)$, proceed as follows to define I_{n+1}: If $J = (a, b)$ then let $NJ = (Na, Nb)$. We claim that there is an M such that

$$\bigcup_{N \geq M} NI_n = (Mc_n, \infty)$$

To see this, we just want to choose M large enough so that $Mc_n < (M+1)c_n < Md_n$ and this happens whenever $M > c_n/(d_n - c_n)$. Pick i large enough and $N \geq M$ so that $J = (a_i, b_i) \cap NI_n$ is non-empty; say it equals (c, d). Let $I_{n+1} = (c/N, d/N)$. By choosing $x \in I_n$ for all $n \in N$ (completeness of the reals), we see that for infinitely many N, $|f(Nx)|$ is greater than ϵ which contradicts the assumption that $f(nx)$ tends to 0 as n tends to ∞.

The second problem we discussed was A1 from last year's contest. One can look at the solution online. The only comments were that the best way to tackle such a problem is to first try a few cases, form a conjecture, check as best you can that the conjecture is correct and then try to write out a proof. The **Putnam moment** here is the use of the pigeonhole principle, the most vanilla version of which is: if you have n pigeons and m holes to put them in where $n > m$ then more than one pigeon winds up in one hole. The way it is applied is that if you have $k > \lceil (n+1)/2 \rceil$ then any way you put n pigeons (numbers) into pigeonholes (boxes) leads to at least two holes with 1 pigeon. If not, then there would be at least $2(k-1)+1$ pigeons which is greater than n. There are a number of places to look for discussions of the PHP; one might start with the wikipedia page.

The final problem we looked at was A4 from last year: Show that $M_n = 10^{10^{10^n}} + 10^{10^n} + 10^n - 1$ is never prime for any $n \in N$. This problem's **Putnam moment** has to do with modular arithmetic; again, the associated wikipedia page is a good place to start if you have never seen this concept before. We checked that in the case that n is odd that this number is divisible by 11. It was left as an exercise to decide what number divides M_n if $n = 2^k m$ where m is odd.

Problem session 3 Let's look at a couple of questions from last year that we didn't get to last week: A2 and B1. I also want to talk a little about how we know limits exist (without actually calculating them!) Here is a good example:

Prove that the following limit exists:

$$\lim_{n \to \infty} 1/n + 1/(n+1) + \ldots + 1/2n$$