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Stationary layered solutions in $\mathbb{R}^2$ for an Allen–Cahn system with multiple well potential

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Abstract. We study entire solutions on $\mathbb{R}^2$ of the elliptic system $-\Delta U + \nabla W(u) = 0$ where $W : \mathbb{R}^2 \to \mathbb{R}^2$ is a multiple-well potential. We seek solutions $U(x_1, x_2)$ which are "heteroclinic," in two senses: for each fixed $x_2 \in \mathbb{R}$ they connect (at $x_1 = \pm\infty$) a pair of constant global minima of W, and they connect a pair of distinct one dimensional stationary wave solutions when $x_2 \to \pm\infty$. These solutions describe the local structure of solutions to a reaction-diffusion system near a smooth phase boundary curve. The existence of these heteroclinic solutions demonstrates an unexpected difference between the scalar and vector valued Allen–Cahn equations, namely that in the vectorial case the transition profiles may vary tangentially along the interface. We also consider entire stationary solutions with a "saddle" geometry, which describe the structure of solutions near a crossing point of smooth interfaces.

1 Introduction

In this paper we study entire solutions $U(x)$ on $\mathbb{R}^2$ to the semilinear elliptic system,

$$-\Delta U + \nabla W(U) = 0 \quad x = (x_1, x_2) \in \mathbb{R}^2, \tag{1.1}$$

where $W : \mathbb{R}^2 \to \mathbb{R}$ is a multiple-well potential: $W(\xi) \geq 0$ attains its global minimum $W(\mathbf{c}_i) = 0$ at a finite number of vectors $\mathbf{c}_1, \ldots, \mathbf{c}_p$. In particular, we seek solutions U which satisfy certain asymptotic conditions imposed for $|x| \to \infty$.

Equation (1.1) arises in the local asymptotic analysis of the reaction-diffusion system,

$$\frac{\partial V}{\partial t} = \varepsilon^2 \Delta V - \nabla W(V), \quad x \in \Omega, \tag{1.2}$$

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for $\Omega \subset \mathbb{R}^2$ a domain. Formal analysis (see Rubinstein, Sternberg & Keller [16], Bronsard & Reitich [4]) suggests that as $\varepsilon \to 0^+$ solutions of (1.2) tend almost everywhere to the minima of W, introducing sharp phase boundaries separating these regions. Equation (1.1) then appears as the first term in the inner expansion about a point lying on the interface.

Intuitively, one might expect that the local behavior of the solution to (1.2) near a smooth point of the interface should resemble that of the scalar Allen-Cahn equation, since locally there are only two phases involved. To be more precise, connecting each pair of minima of W there are one dimensional stationary waves (heteroclinics), and these give special solutions to (1.1) which make the transition between the two phases in the direction orthogonal to the (smooth) phase boundary. However, a more careful analysis of the one dimensional stationary waves reveals a significant difference between the vector-valued equation (1.2) and its scalar version, even along the smooth portion of the interfaces. Namely, there might be several distinct stationary waves connecting a single pair of minima, and in that case it might be possible that the profile of the solution vary tangentially along the transition surface, changing gradually from one stationary wave-form to another along the interface. Expanding such a solution in a neighborhood of the interface, we would see (to first order) a solution to the elliptic system (1.1) in $\mathbb{R}^2$ which converges to two different one dimensional stationary waves in x_1 as $x_2 \to \pm\infty$. We call these special solutions to (1.1) *heteroclinic solutions* in analogy with the classical use of this term in dynamical systems, but in one dimension higher.

We note that in the case where W has three global minima, all three phases may be present in some neighborhood, forcing the interfaces to join at a triple junction. Near such a point the interfaces are no longer smooth, and the local behavior of the solution to (1.2) should be described by a *three layered solution* to (1.1) in $\mathbb{R}^2$, that is it should tend toward each of the three constant minima in a sector of the plane, and towards one-dimensional stationary waves (which connect these minima) across each sector. In [3], Bronsard, Gui & Schatzman established rigorously the existence of such triple–layered solutions in $\mathbb{R}^2$ for (1.1) with a triple-well potential W having the symmetry of an equilateral triangle.

To present the situation more precisely, we need to consider one-dimensional *stationary wave* solutions associated to (1.1). Fix two distinct zeros of W, vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$. For vector-valued functions $z \in [H^1_{loc}(\mathbb{R})]^2$ with $z(t) \to \mathbf{a}$ as $t \to -\infty$ and $z(t) \to \mathbf{b}$ as $t \to +\infty$, define the energy

$$F(z) = \int_{-\infty}^{\infty} \left\{ \frac{1}{2} |z'(t)|^2 + W(z(t)) \right\} dt. \tag{1.3}$$

Standard arguments show that (under reasonable assumptions on W — see below) F attains its infimum in this class at a heteroclinic trajectory connecting $\mathbf{a}$ to $\mathbf{b}$. We assume that W is chosen so that the minimizer is *not unique* (modulo translations in t), but consists of a finite number $k \geq 2$ of geometrically distinct trajectories, $z_1(t), \ldots, z_k(t)$. We choose any two of these elements z_1, z_2, and pose the following problem: find a two-dimensional heteroclinic solution $U(x_1, x_2)$ to (1.1) with conditions imposed as $|x| \to \infty$,

(1.4) $$U(x_1, x_2) \to \mathbf{a} \quad \text{as } x_1 \to -\infty, \text{ uniformly in } x_2;$$

(1.5) $$U(x_1, x_2) \to \mathbf{b} \quad \text{as } x_1 \to +\infty, \text{ uniformly in } x_2;$$

(1.6) $$U(x_1, x_2) \to z_1(x_1) \quad \text{as } x_2 \to +\infty, \text{ uniformly in } x_1;$$

(1.7) $$U(x_1, x_2) \to z_2(x_1) \quad \text{as } x_2 \to -\infty, \text{ uniformly in } x_1.$$

In order to describe our results we first introduce our hypotheses on the potential W. First, W is smooth, with p non-degenerate global minima, and grows rapidly to infinity as $|\xi| \to \infty$:

(1.8) $$W \in C^2(\mathbb{R}^2), \quad W(\mathbf{c}_i) = 0, \quad W(\xi) > 0,\ \xi \neq \mathbf{c}_i, \quad i = 1, \dots, p;$$

(1.9) There exists $\lambda > 0$ such that $D^2 W(\mathbf{c}_i) \geq \lambda I, \quad i = 1, \dots, p;$

(1.10) $$\nabla W(\xi) \cdot \xi \geq 0 \quad \text{for } |\xi| \geq R_0, \text{ some } R_0 > 1.$$

We must also assume that W is symmetric about the perpendicular bisector of the segment connecting the vectors $\mathbf{a}$, $\mathbf{b}$. Without loss of generality, we may assume that the points $\mathbf{b} = (b, 0) = -\mathbf{a}$ lie on the horizontal axis in (U_1, U_2)-space and $b > 0$. With this choice of variables, we assume:

(1.11) $$W(\gamma(\xi)) = W(\xi), \quad \text{where } \gamma(\xi_1, \xi_2) = (-\xi_1, \xi_2).$$

Note, however, that we do *not* need to impose symmetry in the ξ_2-direction. The symmetry hypothesis (1.11) is a technical assumption which we use to eliminate loss of compactness via translations in the x_1-direction. Without symmetry we cannot verify that our minimizing sequences (even after suitable translation) attain the desired asymptotic conditions (1.4)–(1.7). Although we believe that this restriction is only technical, we do not know if this problem presents some new and unfamiliar kind of loss of compactness. (See the discussion in Sect. 5.)

We define an admissible set for the energy $F(z)$ defined in (1.3),

(1.12) $$\mathcal{S}_{ab} := \{z(t) = (z^1(t), z^2(t)) \in \left(H^1_{loc}(\mathbb{R})\right)^2 : \lim_{t \to -\infty} z(t) = \mathbf{a}, \\ \lim_{t \to +\infty} z(t) = \mathbf{b},\ z(-t) = (-z^1(t), z^2(t))\}.$$

Note that symmetry of $z \in \mathcal{S}_{ab}$ removes the degeneracy due to the translation invariance of the functional F. We denote by e_{ab} the minimum energy required to connect wells $\mathbf{a}$ and $\mathbf{b}$ by a symmetric heteroclinic orbit:

(1.13) $$e_{ab} := \min\{F(z) :\ z(t) \in \mathcal{S}_{ab}\}.$$

We remark that the energy-minimizing connecting orbits $z(t)$ may be identified with minimizing geodesics in a Riemannian metric determined by the potential W (see [4], [18].) In this sense, our next hypothesis is a sort of strict triangle inequality:

(1.14) $$e_{ab} < e_{ac} + e_{cb}, \quad \text{for any zero } \mathbf{c} \neq \mathbf{a}, \mathbf{b} \text{ of } W.$$

As we will see later, this condition ensures the existence of a one dimensional stationary wave connecting the minima **a** and **b**, and that the optimal path does not pass by a third minimum, **c**. Note that the right-hand side of (1.14) is always at least as large as e_{ab}, so we are merely eliminating the possibility that the two quantities are in fact equal.

Finally, we may state an explicit theorem in the case where W admits *exactly two* geometrically distinct energy-minimizing heteroclinic trajectories $z_1(t), z_2(t)$ which connect **a** to **b**. We refer the reader to Theorem 3.3 for the more general case where $k \geq 2$.

Theorem 1.1 (Case $k = 2$**)** *Assume that* W, $\mathbf{a}, \mathbf{b}$ *satisfy hypotheses (1.8)–(1.11) and (1.14). In addition, suppose that* $F(z)$ *attains its minimum at exactly two curves* $z_1, z_2 \in \mathscr{S}_{ab}$. *Then there exists an entire solution* $U(x_1, x_2)$ *of (1.1) in* $\mathbb{R}^2$ *satisfying the conditions (1.4)–(1.7).*

Although there is a canonical energy associated to the equation (1.1), $\int_{\mathbb{R}^2} \frac{1}{2}|\nabla U|^2 + W(U)\, dx$, the solutions which we seek must necessarily have infinite energy. Motivated by the work of P. Rabinowitz on heteroclinic solutions for Hamiltonian ODE systems (see [13] and [15],) we show that the heteroclinic solutions of (1.1) may be obtained as global minimizers of a "renormalized" energy (3.1). However, this energy is not coercive on any natural space, and there is no obvious way to obtain the heteroclinic solution via the direct method. We obtain these solutions via approximation: solving boundary-value problems, obtaining *a priori* bounds, and passing to the limit, first as the rectangular regions approach infinite strips and then as the width of the infinite strips tends to infinity. The global variational framework provides energy estimates which are necessary to verify that the limiting solution does indeed exhibit the desired asymptotic shape.

To compare our problem with the more familiar scalar case, consider the typical example $W_0(u) = \frac{1}{4}(u^2 - 1)^2$, $u \in \mathbb{R}$, with with global minima ± 1. The family $\zeta(t) = \tanh([t - t_0]/\sqrt{2})$ ($t_0 \in \mathbb{R}$ any constant) describe heteroclinic trajectories for the ODE

$$-u'' + W_0'(u) = 0. \tag{1.15}$$

In fact, these solutions minimize the associated energy functional

$$F_0(u) = \int_{-\infty}^{\infty} \left[\frac{1}{2}|u'(t)|^2 + W_0(u(t)) \right] dt$$

among functions $u \in H^1(\mathbb{R})$ with $u(t) \to \pm 1$ as $t \to \pm\infty$, and they are known to be the only stable equilibria of the corresponding parabolic system in one dimension (see eg. [5].) In this scalar setting, De Giorgi [7] has conjectured that *any* entire solution $u : \mathbb{R}^n \to \mathbb{R}$ of $-\Delta u + W_0'(u) = 0$ which connects ± 1 at $x_1 = \pm\infty$ is in fact one-dimensional: its level sets are hyperplanes, and by a suitable rotation of $\mathbb{R}^n$ is of the form $u(x) = z(x_1)$ for some heteroclinic solution to (1.15). Modica & Mortola [10] have shown that De Giorgi's conjecture holds in dimension $n = 2$, but only under the additional hypothesis that the solution's level curves be Lipshitz graphs. The higher dimensional case is entirely open, although some evidence of the validity of the conjecture is provided in the paper by Caffarelli, Garofalo & Segala [5]. Our results show that this conjecture does *not* generally hold for vector-valued equations of bistable type.

We remark that heteroclinic solutions to PDEs have been considered by others, in somewhat different settings. Solutions of semilinear elliptic equations which are homoclinic to zero were studied by Coti-Zelati & Rabinowitz [6] (for periodic coefficients) and by Alama & Li [1] (with asymptotic periodicity.) Bates & Ren [2] have shown the existence of heteroclinic solutions for a high order scalar equation (1.15) in periodic strip-like domains. Rabinowitz [15] has also considered solutions of the semilinear scalar equation,

$$-\Delta u = g(x,y,u), \quad x \in \Omega, \quad y \in \mathbb{R}, \tag{1.16}$$

in cylindrical domains $\Omega \times \mathbb{R}$. Under the hypothesis that $g(x,y,u)$ is periodic in y,u, conditions are given in [15] which ensure the existence of solutions which approach as $y \to \pm\infty$ different periodic solutions of (1.16).

Finally, in Section 6 we consider "saddle solutions" of equation (1.1) in $\mathbb{R}^2$. Given a pair $\mathbf{a}, \mathbf{b}$ of minima of W, we seek a solution U which satisfies

$$\lim_{x_2 \to +\infty} U(x_1,x_2) = z(x_1), \qquad \lim_{x_2 \to -\infty} U(x_1,x_2) = z(-x_1),$$
$$\lim_{x_1 \to +\infty} U(x_1,x_2) = z(x_2), \qquad \lim_{x_1 \to -\infty} U(x_1,x_2) = z(-x_2),$$

for some one-dimensional heteroclinic trajectory z connecting $\mathbf{a}$ to $\mathbf{b}$. Such a solution describes the local behavior of a solution to the reaction-diffusion system (1.2) at a point where interfaces cross. The resulting configuration of two phases alternating around a cross-shaped interface is believed to be highly unstable, as it represents a singularity in the flow by curvature obtained as the limiting problem ($\varepsilon \to 0$) for (1.2). Stationary saddle solutions were obtained by [8] for the scalar Allen-Cahn equation, by means of a sub- and super-solution method, and their stablility was studied by Schatzman [17]. As is well-known, such monotonicity techniques are not effective in studying systems of equations, and hence their result does not extend to our vector-valued equation (1.1). In Sect. 6 we derive the existence of a saddle solution under similar hypotheses to those listed above for the heteroclinic case. When restricted to the scalar equation case, our result improves the existence result in [8]. On the other hand, for the vector equation case, our result cannot permit a zero of W to lie on the axis of symmetry between the chosen zeros $\mathbf{a}$ and $\mathbf{b}$, and therefore the existence of saddle solutions for the three-welled potential of [4], [3] remains an open question.

2 The one dimensional problem

We begin with a brief discussion of the associated one dimensional problem,

$$-z''(t) + \nabla W(z) = 0, \qquad \lim_{t \to -\infty} z(t) = \mathbf{a}, \ \lim_{t \to +\infty} z(t) = \mathbf{b}. \tag{2.1}$$

In particular, we are interested in those solutions of (2.1) which minimize the energy $F(\cdot)$ defined in (1.3) in the class of curves $\mathscr{S}_{ab}$ (defined in 1.12). Following [3], we can show that under the conditions (1.8)–(1.11) and (1.14) for W,

$$e_{ab} = \inf\{F(z) : \ z \in \mathscr{S}_{ab}\}$$

is attained, and its minimizers satisfy (2.1). We sketch the proof below, and refer the reader to more general existence results of Rabinowitz [13], [14].

First we state a basic energy estimate, which will be useful throughout this paper:

Lemma 2.1 *Suppose* $v \in [H^1([L_1, L_2])]^2$ *and* $|v(\pm L_i)| \le R, i = 1, 2$. *Then,*

$$\int_{L_1}^{L_2} \frac{1}{2}|v'(t)|^2 + W(v)\,dt \ge e_{ab} - C_0 \frac{\left(|v(L_1) - \mathbf{a}|^2 + |v(L_2) - \mathbf{b}|^2\right)}{2},$$

where $C_0 = 1 + \frac{1}{3}\max\{|D^2W(\xi)| : \ |\xi| \le R + |\mathbf{a}|\}$.

The proof of Lemma 2.1 is similar to that of Lemma 2.6 in [3]. We only need to construct a function

$$\tilde{v}(t) = \begin{cases} v(t), & \text{if } L_1 \le t \le L_2; \\ v(L_1)(t - L_1 + 1) + \mathbf{a}(L_1 - t), & \text{if } L_1 \le t \le L_1; \\ v(L_2)(L_2 + 1 - t) + \mathbf{b}(t - L_2), & \text{if } L_2 \le t \le L_2 + 1; \\ \mathbf{a}, & \text{if } t < L_1 - 1; \\ \mathbf{b}, & \text{if } t > L_2 + 1. \end{cases}$$

Then $F(\tilde{v}) \ge e_{ab}$ and Lemma 2.1 follows by straightforward computations.
♢

We now return to the existence of minimizing trajectories for F in $\mathscr{S}_{ab}$. Let

$$\varphi(t) = \begin{cases} \mathbf{a}, & \text{if } t < -1; \\ \mathbf{b}, & \text{if } t > 1; \\ \frac{1}{2}[(\mathbf{b} - \mathbf{a})t + (\mathbf{b} + \mathbf{a})] & \text{if } -1 \le t \le 1. \end{cases}$$

Suppose that $\{v_n(t)\} \in \mathscr{S}_{ab}$ is a minimizing sequence for $F(z)$. By (1.11), (1.10) we can assume, without loss of generality, that $|v_n(t)| < R_0, \forall t \in \mathbb{R}$ and that the first component of $v_n(t)$ is nonpositive for $t \in \mathbb{R}$ and $v_n(t) - \varphi(t) \in (H^1(\mathbb{R}))^2$ (see, e.g. [3] for details). Using (1.14) and Lemma 2.1, we know that for every minimum point $\mathbf{c} \neq \mathbf{a}, \mathbf{b}$ of W,

$$|v_n(t) - \mathbf{c}| \ge \delta > 0, \quad \forall t \in \mathbb{R}$$

when n is large enough, where δ is a positive constant. By (1.9), we have

$$W(v) \ge \kappa \min\{|v - \mathbf{a}|^2, |v - \mathbf{b}|^2\}, \tag{2.2}$$

$$\forall v \in \{v | |v| \le R_0, |v - \mathbf{c}| \ge \delta, \ \text{for } \mathbf{c} \neq \mathbf{a}, \mathbf{b} \text{ a zero of } W.\} \tag{2.3}$$

where κ is a positive constant depending only on W, δ. Since $F(v_n) \to e_{ab}$, as $n \to \infty$, the above inequality leads to the boundedness of $v_n - \varphi$ in $(H^1(\mathbb{R}))^2$ and consequently the weak convergence of $v_n - \varphi$ to $v - \varphi$ in $(H^1(\mathbb{R}))^2$. Furthermore, $F(v) \leq e_{ab}$ and $v_n(t)$ converges to $v(t)$ uniformly in $t \in [-L, L]$ for any fixed L. Then $|v(t) - \mathbf{c}| \geq \delta > 0, \forall t \in \mathbb{R}$ where $\mathbf{c} \neq \mathbf{a}, \mathbf{b}$ is any minimum point of W. By symmetry of v, it is easy to see that $v \in \mathscr{S}_{ab}$. This proves the existence of minimizer of $F(z)$ in $\mathscr{S}_{ab}$.

We note that if (1.14) is *not* assumed, there might *not* exist a minimizer for $F(z)$ in $\mathscr{S}_{ab}$. Even assuming (1.14) holds, the minimizer may not be unique. We denote the set of minimizers of F in $\mathscr{S}_{ab}$ by $\mathscr{Z} = \mathscr{Z}_{ab} = \{z_1, \ldots, z_k\}$ and assume that there is non-uniqueness:

$$2 \leq \text{Card}(\mathscr{Z}) = k < \infty. \tag{2.4}$$

Remark 2.2 Note that solutions of (2.1) are translation invariant, but the symmetry condition incorporated in $\mathscr{S}_{ab}$ fixes a representative. Generically, we expect the set $\mathscr{Z}$ to be finite (see Proposition 2.1 in [3].)

Following are some basic properties of the minimizers $z \in \mathscr{Z}$, many of which are already established in [3].

Proposition 2.3 *Assume hypotheses (1.8)–(1.11). Any $z \in \mathscr{Z}$ is a C^3 embedded curve in $\mathbb{R}^2$. In addition, the image curves corresponding to distinct elements z_i, z_j $(i \neq j)$ are nonintersecting. Moreover, minimizers z exhibit exponential decay: there exists constants $C > 0$ and $\mu > 0$ such that*

$$|z(t) - \mathbf{a}| + |z'(t)| \leq C \exp\{-\mu t\}, \quad t \in \mathbb{R}. \tag{2.5}$$

Note that fact that image curve do not intersect themselves or other curves is a consequence of energy minimization and is not necessarily true for non-minimizing solutions of (2.1). We remark that the results of Proposition 2.3 hold without the symmetry hypothesis (1.10), with the definitions of $\mathscr{S}_{ab}$ and $\mathscr{Z}$ modified appropriately.

Proof. The regularity of minimizers z is standard. Self-intersections may be ruled out using the fact that each z is an energy minimizer. Indeed, if $z(t_1) = z(t_2)$ for $t_1 \neq t_2$ then a new admissible curve $\tilde{z}(t) \in \mathscr{S}$ may be constructed so that it excises the loop $\{z(t) : t_1 < t < t_2\}$ and has strictly smaller energy. Intersections between distinct elements of $\mathscr{Z}$ may be ruled out in a similar way. First note that the symmetry condition $z(-t) = (-z^1(t), z^2(t))$ forces minimizers to cross the vertical $(z^1 = 0)$ axis exactly once, and with $t = 0$, since otherwise $z(t)$ would have a self-intersection. If $z_1, z_2 \in \mathscr{Z}$ cross, we may then assume that $z_1(t_1) = z_2(t_2)$ with $t_1, t_2 \geq 0$, and that the energy of z_1 on $[t_1, \infty)$ is smaller than or equal to the energy of z_2 on $[t_2, \infty)$, i.e.

$$\int_{t_1}^{\infty} \left(\frac{1}{2}|z_1'(t)|^2 + W(z_1(t))\right) dt \leq \int_{t_2}^{\infty} \left(\frac{1}{2}|z_2'(t)|^2 + W(z_2(t))\right) dt.$$

Now we construct a new admissible curve $\tilde{z}(t) \in \mathscr{S}_{ab}$ by patching:

$$\tilde{z}(t) = \begin{cases} z_1(t - t_2 + t_1), & \text{if } t \geq t_2; \\ z_2(t), & \text{if } t_2 > t \geq 0; \end{cases}$$

and $z(-t) = (-z^1(t), z^2(t))$ for $t < 0$. Since $F(z_1) = F(z_2) = e_{ab}$, we have

$$\int_0^{t_1} \left(\frac{1}{2}|z_1'(t)|^2 + W(z_1(t))\right) dt \geq \int_0^{t_2} \left(\frac{1}{2}|z_2'(t)|^2 + W(z_2(t))\right) dt.$$

Therefore we derive $F(\tilde{z}) \leq e_{ab}$, and $\tilde{z}(t)$ is a minimizer. By regularity of minimizers, z_1 and z_2 must have the same derivatives at the cross point. By the uniqueness of solutions to initial value problems for ordinary differential equations, we conclude that z_1 and z_2 must be identical.

The exponential decay estimate (2.5) is proven in [3].

$\diamondsuit$

Our point of view is that a candidate $U(x_1, x_2)$ for a solution to (1.1) in $\mathbb{R}^2$ is, for each fixed value of x_2, an element of $\mathscr{S}_{ab}$, and hence may be compared with the minimizers $z \in \mathscr{Z}$ via estimates on the one-dimensional energy values

$$F(U(\cdot, x_2)) = \int_{-\infty}^{+\infty} \left[\frac{1}{2}\left|\frac{\partial U}{\partial x_1}\right|^2 + W(U(x_1, x_2))\right] dx_1.$$

The following lemma shows that the one-dimensional energy is strongly coercive on the set $\mathscr{S}_{ab}$: if the energy of a function v is close to the minimum, then it must be close to a minimizer.

Lemma 2.4 *For any $\varepsilon > 0$ there exists $\delta > 0$ such that if $v \in \mathscr{S}_{ab}$ and $F(v) \leq e_{ab} + \delta$, then there exists $z_\alpha \in \mathscr{Z}$ such that*

$$\|v - z_\alpha\|_{H^1 \cap L^\infty(\mathbb{R})} < \varepsilon.$$

It suffices to show that if $v_n \in \mathscr{S}_{ab}$ and $\lim_{n\to\infty} F(v_n) = e_{ab}$, then there exists a subsequence of v_n, which we still denote by v_n, and $z_\alpha \in \mathscr{Z}$ such that

$$\|v_n - z_\alpha\|_{H^1 \cap L^\infty(\mathbb{R})} \to 0.$$

First, we show the convergence in $L^\infty(\mathbb{R})$.

Claim 1. v_n is bounded in $L^\infty(\mathbb{R})$.

Suppose not, then there exists t_n such that $|v_n(t_n)| \to \infty$ as $n \to \infty$. By the continuity of $v_n(t)$ in t and the fact that $v_n(t) \to \mathbf{a}$ as $t \to -\infty$, for n sufficiently large there exists $s_n < t_n$ such that $|v_n(s_n)| = R_0$ and $|v_n(t)| \geq R_0, \forall t \in (s_n, t_n)$. By (1.8) and (1.11), we have

$$W(\xi) \geq m_0, \quad \text{for} \quad |\xi| \geq R_0.$$

where $m_0 > 0$ is a constant.

Then

$$\begin{aligned} F(v_n) &\geq \int_{s_n}^{t_n} \frac{1}{2}|v_n'(t)|^2 + W(v_n(t))dt \\ &\geq \int_{s_n}^{t_n} \sqrt{m_0}|v_n'(t)|dt \geq \sqrt{m_0}|v_n(t_n) - v_n(s_n)| \\ &\geq \sqrt{m_0}(|v_n(t_n)| - R_0). \end{aligned}$$

Letting $n \to \infty$ we get a contradition. This proves the claim.

Let $m(r) := \min\{W(\xi) : |\xi - \mathbf{a}| \geq r, |\xi - \mathbf{b}| \geq r\}$ for $r > 0$. For any given $\varepsilon > 0$, if we choose $L = \frac{4e}{m(\varepsilon^2)} > 0$, then there exists $t_n \in [-L, -L/2]$ such that either $|v_n(t_n) - \mathbf{a}| \leq \varepsilon^2$ or $|v_n(t_n) - \mathbf{b}| \leq \varepsilon^2$ for n sufficiently large. Since otherwise, we would have

$$F(v_n) \geq \int_{-L}^{-L/2} \frac{1}{2}|v_n'(t)|^2 + W(v_n(t))dt \geq m(\varepsilon^2)L/2 \geq 2e_{ab}.$$

This is a contradiction when $n \to \infty$.

We may assume that ε is sufficiently small, then we have indeed $|v_n(t_n) - \mathbf{a}| \leq \varepsilon^2$, thanks to Lemma 2.1.

Claim 2. There exists $N_0 = N_0(\varepsilon)$ such that when $n > N_0$, we have

$$|v_n(t) - \mathbf{a}| \leq \varepsilon, \quad \forall t \leq -L.$$

Suppose otherwise, for all n sufficiently large there exist $s_n < \sigma_n \leq -L$ such that

$$|v_n(s_n) - \mathbf{a}| = \varepsilon/2, \quad |v_n(\sigma_n)| = \varepsilon, \quad \text{and} \quad \varepsilon \geq |v_n(t) - \mathbf{a}| \geq \varepsilon/2, \quad \forall t \in [s_n, \sigma_n].$$

Then by Lemma 2.1 and (1.9), we have

$$\begin{aligned} F(v_n) &\geq 2\int_{s_n}^{\sigma_n} \frac{1}{2}|v_n'(t)|^2 + W(v_n(t))dt + \int_{t_n}^{-t_n} \frac{1}{2}|v_n'(t)|^2 + W(v_n(t))dt \\ &\geq 2\int_{s_n}^{\sigma_n} \frac{1}{2}\sqrt{\lambda}|v_n'(t)| \cdot |v_n(t) - \mathbf{a}|dt + e_{ab} - C_0\varepsilon^4 \\ &\geq e_{ab} + C\varepsilon^2 - C_0\varepsilon^4 \end{aligned}$$

where $C = \sqrt{\lambda}/4 > 0$. Letting $n \to \infty$, we have a contradiction when ε is sufficiently small. This proves Claim 2.

Now it is easy to see that v_n is equicontinuous in $\mathbb{R}$, and that there exists a subsequence of v_n, which we still denote by v_n, converging uniformly on $\mathbb{R}$ to $z \in \mathscr{S}$. Moreover, we have $F(z) \leq e_{ab}$ and then $z = z_\alpha \in \mathscr{Z}$ for some α.

To show convergence in $H^1(\mathbb{R})$, let $\varepsilon > 0$ be given, and then let L, N_0 be chosen so that

(2.6) $$|v_n(x) - \mathbf{a}| < \varepsilon, \quad x < -L, \quad \text{and} \quad |v_n(x) - \mathbf{b}| < \varepsilon, \quad x > L,$$

(2.7) $$F(v_n) - e_{ab} < \varepsilon^2 \quad \text{for } n \geq N_0,$$

(2.8) $$\int_{-\infty}^{-L} \frac{1}{2}|z'(t)|^2 + W(z(t))\,dt \leq \varepsilon^2.$$

From hypothesis (1.9), (2.6), and (2.8), when ε is sufficiently small we obtain:

$$\int_{-\infty}^{-L} \frac{1}{2}|z'(t)|^2 + \frac{\lambda}{4}|z(t)|^2\,dt \leq \varepsilon^2.$$

By Lemma 2.1 and hypothesis (1.9) we also have:

$$\int_{-\infty}^{-L} \frac{1}{2}|v_n'(t)|^2 + \frac{\lambda}{4}|v_n(t)|^2\, dx_1 \le \int_{-\infty}^{-L} \frac{1}{2}|v_n'(t)|^2 + W(v_n(t))\, dt \le C\varepsilon^2.$$

In particular, we have:

$$\int_{-\infty}^{-L} \frac{1}{2}|v_n' - z'|^2 + \frac{\lambda}{4}|v_n - z|^2\, dt < C\varepsilon^2.$$

(By symmetry, identical estimates hold over the interval $[L, \infty)$.)

With L fixed (as above), the convergence $v_n \to z$ in L^∞ implies that for $n \ge N_1 \ge N_0$,

$$\int_{-L}^{L} |v_n - z|^2 + |W(v_n) - W(z)|\, dt < \varepsilon^2.$$

In particular, we have $v_n - z \to 0$ in $L^2(\mathbb{R})$. In view of (2.7) and the above estimate, choosing n sufficiently large we obtain:

$$\left| \int_{-L}^{L} \left(|v_n'|^2 - |z'|^2 \right)\, dt \right| < \varepsilon^2.$$

By an integration by parts,

$$\left| \int_{-L}^{L} (v_n' - z') \cdot z'\, dt \right| \le 2|(v_n - z)(L)|\, |z(L)| + \left[\int_{-L}^{L} |v_n - z|^2\, dt \right]^{1/2} \times \left[\int_{-L}^{L} |z''|^2\, dt \right]^{1/2}.$$

Since z'' is integrable, we may conclude that for n sufficiently large

$$\int_{-L}^{L} |v_n - z|^2\, dt = \int_{-L}^{L} \left(|v_n'|^2 - |z'|^2 \right)\, dt - 2\int_{-L}^{L} (v_n' - z') \cdot z'\, dt < \varepsilon^2.$$

In conclusion, $v_n - z \to 0$ in $H^1(\mathbb{R})$, as desired.
♢

3 The variational formulation

We now turn to the two dimensional problem. Throughout the rest of the paper we fix two distinct minima $\mathbf{a}, \mathbf{b}$ of W, with $\mathbf{a} = -\mathbf{b}$ and $\mathbf{b} = (b, 0)$, and we assume that the strict triangle inequality (1.14) holds for this choice of $\mathbf{a}, \mathbf{b}$. Where there is no ambiguity we write $e = e_{ab}$ in (1.13). Given two distinct elements $z_1, z_2 \in \mathscr{Z}$, we define a class of admissible functions by

$$\mathcal{M}_{1,2} = \left\{ \begin{array}{c} u \in \left(H^1_{loc}(\mathbb{R}^2)\right)^2 \cap \left(C^0(\mathbb{R}^2)\right)^2 : \\ u \circ \gamma(x) = \gamma \circ u(x), \\ |u(x) - \varphi_{1,2}(x)| \to 0 \text{ as } |x| \to \infty \end{array} \right\}$$

where

$$\varphi_{1,2}(x) = \begin{cases} z_1(x_1), & \text{if } x_2 \geq 1; \\ z_2(x_1), & \text{if } x_2 \leq -1, \\ z_1(x_1)\frac{1+x_2}{2} + z_2(x_1)\frac{1-x_2}{2}, & \text{if } -1 < x_2 < 1, \end{cases}$$

and $\gamma(\xi) = (-\xi_1, \xi_2)$. Recall that all elements of $\mathcal{Z}$ are minimizers with equal one-dimensional energy $e = F(z_\alpha) > 0$, $\alpha = 1, \dots, k$. We define a renormalized two-dimensional energy by subtracting energy e from each horizontal strip:

$$\mathcal{E}(U) = \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} \frac{1}{2}|\nabla U|^2 + W(U)\, dx_1 - e \right] dx_2, \quad U \in \mathcal{M}_{1,2}. \tag{3.1}$$

(We remark that the idea of decomposing an unbounded domain into strips and subtracting off a "base" quantity of energy on each strip has been introduced by Rabinowitz [13] for finding trajectories of Hamiltonian systems which are heteroclinic to periodic solutions.) By renormalizing we expect our solutions to have finite energy $\mathcal{E}$:

Proposition 3.1

$$0 < \sigma_{1,2} = \inf_{U \in \mathcal{M}_{1,2}} \mathcal{E}(U) < +\infty.$$

Proof. By taking $U = \varphi_{1,2} \in \mathcal{M}_{1,2}$, it is clear that $\sigma_{1,2} < +\infty$. To see this, note that the outer integral in $\mathcal{E}$ is taken over the finite interval $[-1, 1]$, and each term is then finite due to the exponential decay (2.5) of any $z \in \mathcal{Z}$ to either $\mathbf{a}$ or $\mathbf{b}$.

We will now show that $\sigma_{1,2} > 0$. Fix a finite interval in x_1, $[-L, L]$, and set

$$\rho = \min_{\alpha \neq \beta} \|z_\alpha - z_\beta\|_{L^\infty([-L,L])} > 0$$

(since the curves z_α do not intersect.) By Lemma 2.4 we may choose $\delta_0 > 0$ such that whenever $v \in \mathcal{S}$ satisfies $F(v) \leq e + \delta_0$, then there exists $z \in \mathcal{Z}$ with $\|v - z\|_{L^\infty(\mathbb{R})} < \rho/8$.

Now fix $U \in \mathcal{M}_{1,2}$. Since $U(x_1, x_2) \to z_1(x_1)$ uniformly in x_1 as $x_2 \to +\infty$, eventually the trajectory $U(\cdot, x_2) \in \mathcal{Z}$ must leave an $L^\infty[-L, L]$-neighborhood of z_2: there exists $M_1 \in \mathbb{R}$ such that

$$\|U(\cdot, x_2) - z_2\|_{L^\infty[-L,L]} \geq \frac{\rho}{4} \quad \text{for all } x_2 \geq M_1 \quad \text{and}$$
$$\|U(\cdot, M_1) - z_2\|_{L^\infty[-L,L]} = \frac{\rho}{4}.$$

Define also

$$M_2 = \min\{x_2 > M_1 : \ \|U(\cdot, x_2) - z_\alpha\|_{L^\infty[-L,L]} \leq \frac{\rho}{8} \text{ for some } z_\alpha \in \mathcal{Z}.\}.$$

Indeed, $U \in \mathscr{M}_{1,2}$ means that eventually $U(\cdot, x_2)$ is L^∞-close to z_1, but we allow for the possibility that it first passes near some other element $z_\alpha \in \mathscr{Z}$. In particular, for any $x_1 \in \mathbb{R}$ we have (for appropriately chosen $z_\alpha \in \mathscr{Z}$),

$$\begin{aligned} |U(x_1, M_2) - U(x_1, M_1)| \ &\geq \ |z_2(x_1) - z_\alpha(x_1)| - |U(x_1, M_2) - z_\alpha(x_1)| \\ &\quad -|U(x_1, M_1) - z_2(x_1)| \geq \frac{\rho}{2}. \end{aligned}$$

Furthermore, we observe that Lemma 2.4 and the definition of δ_0 imply that:

$$\inf_{x_2 \in (M_1, M_2)} F(U(\cdot, x_2)) \geq e + \frac{\delta_0}{2}.$$

(Note that M_1, M_2 may depend on the function U.)

Now we may estimate the energy of U:

$$\begin{aligned} (3.2) \quad \mathscr{E}(U) \ &\geq \ \int_{M_1}^{M_2} [F(U(\cdot, x_2)) - e] \, dx_2 + \int_{M_1}^{M_2} \int_{-\infty}^{+\infty} \frac{1}{2} |U_{x_2}|^2 \, dx_1 \, dx_2 \\ &\geq \ \frac{\delta_0}{2}(M_2 - M_1) + \frac{1}{2} \int_{-L}^{L} \left[\frac{\left(\int_{M_1}^{M_2} |U_{x_2}| \, dx_2 \right)^2}{\int_{M_1}^{M_2} 1 \, dx_2} \right] dx_1 \\ &\geq \ \frac{\delta_0}{2}(M_2 - M_1) + \frac{L}{M_2 - M_1} \min_{x_1 \in [-L, L]} |U(x_1, M_2) - U(x_1, M_1)|^2 \\ &\geq \ \frac{\delta_0}{2}(M_2 - M_1) + \frac{L\rho^2}{4(M_2 - M_1)} \geq \rho \sqrt{\delta_0 L/2} > 0. \end{aligned}$$

This inequality holds for any $U \in \mathscr{M}_{1,2}$, hence we conclude $\sigma_{1,2} \geq \rho\sqrt{\delta_0 L/2} > 0$.

♢

We next show that minima of the renormalized energy $\mathscr{E}$ are indeed solutions of our PDE:

Proposition 3.2 *Suppose that $U \in \mathscr{M}_{1,2}$ attains the minimum of $\mathscr{E}$,*

$$\mathscr{E}(U) = \inf_{V \in \mathscr{M}_{1,2}} \mathscr{E}(V).$$

Then, $U \in C^{2,\alpha}(\mathbb{R}^2)$ satisfies (1.1) with the heteroclinic conditions (1.4)–(1.7).

Proof. Let $\psi = (\psi_1(x), \psi_2(x)) \in C_0(\mathbb{R}^2; \mathbb{R}^2)$ be given, and choose $M > 0$ such that $\operatorname{supp} \psi \subset [-M, M]^2$. Then, $U + t\psi \in \mathscr{M}_{1,2}$ and the functions $W(U + t\psi)$ are continuous and uniformly bounded in $[-M, M]^2$ for $|t| < 1$. Then,

$$\begin{aligned} 0 \ &\leq \ \mathscr{E}(U + t\psi) - \mathscr{E}(U) \\ &= \int_{-M}^{M} \int_{-M}^{M} \left[\frac{|\nabla(U + t\psi)|^2}{2} - \frac{|\nabla U|^2}{2} + W(U + t\psi) - W(U) \right] dx_2 \, dx_1 \\ &= \int_{-M}^{M} \int_{-M}^{M} \left[\sum_{i,j=1}^{2} \nabla U_i \cdot \nabla \psi_j + \nabla W(U) \cdot \psi \right] dx_1 \, dx_2 + o(t). \end{aligned}$$

In the limit, we have

$$\int_{\mathbb{R}^2} \left[\sum_{i,j=1}^{2} \nabla U_i \cdot \nabla \psi_j + \nabla W(U) \cdot \psi \right] dx = 0$$

for any $\psi \in C_0^\infty(\mathbb{R}^2; \mathbb{R}^2)$. Then, taking test functions of the form: $\psi = (\psi_1, 0)$ and $\psi = (0, \psi_2)$ in the above equality, we obtain each equation in the system (1.1) in weak form. By a standard bootstrap argument we may conclude that U is a strong solution in $C^{2,\alpha}$.
$\diamondsuit$

We have shown that $\mathscr{E}$ is bounded below on the class $\mathscr{M}_{1,2}$, and that minima yield solutions to our equation (1.1) with the desired asymptotic behavior. Obviously, we would be finished if we could show that the minimum of $\mathscr{E}$ were attained in $\mathscr{M}_{1,2}$. The most obvious tool in this situation is the "direct method", namely to show compactness of minimizing sequences for $\mathscr{E}$. One obstacle to compactness is translation invariance: sequences in $\mathscr{M}_{1,2}$ may have weak limits which no longer belong to $\mathscr{M}_{1,2}$, and hence do not satisfy our boundary conditions for U at infinity. However, a greater problem is the fact that $\mathscr{E}$ does not seem to be coercive on $\mathscr{M}_{1,2}$, and hence the fact that minimizers have bounded energy $\mathscr{E}$ does not yield an estimate sufficient for weak compactness in any reasonable space.

Our method will be one of approximation: we will seek a minimizer as a limit of minimizers in strips of finite length. The fact that the solution thus obtained is a minimizer will enable us to eliminate certain pathological cases when taking the limit.

It will be necessary for our method to choose asymptotic states z_1, z_2 which are not only one-dimensional minimizers, but also satisfy another strict triangle inequality for their two-dimensional energy $\mathscr{E}$,

$$\sigma_{1,2} \neq \sigma_{1,\alpha} + \sigma_{\alpha,2} \quad \text{for any } \alpha \neq 1, 2. \tag{3.3}$$

Note that z_1, z_2 satisfy (3.3) if, for example, their two-dimensional energy $\sigma_{1,2}$ is minimal among all pairs of distinct elements of $\mathscr{Z}$. Our general theorem then reads:

Theorem 3.3 *Suppose W, $\mathbf{a}, \mathbf{b}$ satisfy the hypotheses (1.8)–(1.11), (1.14), and (2.4). If $z_1 \neq z_2 \in \mathscr{Z}$ and (3.3) holds, then $\mathscr{E}$ attains its minimum value in the class $\mathscr{M}_{1,2}$ at $U \in \mathscr{M}_{1,2}$. Furthermore, $U \in C^{2,\alpha}(\mathbb{R}^2)$ for $0 < \alpha < 1$ and U satisfies (1.1) in $\mathbb{R}^2$ together with the asymptotic conditions (1.4)–(1.7).*

Of course, if $k = 2$ (i.e., the one-dimensional problem admits exactly two minimizing trajectories), then Theorem 3.3 reduces to Theorem 1.1 stated in the introduction.

4 Infinite strips

In this section we consider solutions to equation (1.1) in semi-infinite strips,

$$T_L = \{x = (x_1, x_2) : \ |x_2| \le L\},$$

satisfying boundary conditions given by distinct elements $z_1, z_2 \in \mathscr{Z}$ on the edges $x_2 = \pm L$.

To solve the problem in strips, we approximate the strips by boxes $R_{L,M} = [-M, M] \times [-L, L]$, where we may easily solve the boundary value problem for (1.1). First, we truncate the one-dimensional solutions z_i, $i = 1, 2$:

$$\varphi_M^i(x_1) = \begin{cases} z_i(x_1), & \text{if } |x_1| \le M - 1; \\ z_i(-M+1)[x_1 + M] + \mathbf{a}[-M + 1 - x_1], & \text{if } -M \le x_1 \le -M + 1; \\ z_i(M-1)[-x_1 + M] + \mathbf{b}[-M + 1 + x_1], & \text{if } M - 1 \le x_1 \le M. \end{cases}$$

Then, we introduce a class of admissible functions incorporating the desired boundary conditions on $R_{L,M}$: let

$$\Phi_{L,M}(x) = \varphi_M^1(x_1)\left(\frac{x_2 + L}{2L}\right) + \varphi_M^2(x_1)\left(\frac{-x_2 + L}{2L}\right)$$

for $x = (x_1, x_2) \in R_{L,M}$, and

$$\mathscr{M}_{1,2}^{L,M} = \left\{U : \ U - \Phi_{L,M} \in \left(H_0^1(R_{L,M})\right)^2, \ U \circ \gamma(x) = \gamma \circ U(x)\right\}.$$

Finally, we introduce an energy on $\mathscr{M}_{1,2}^{L,M}$,

$$E_{L,M}(U) = \int_{T_{L,M}} \frac{1}{2}|\nabla U|^2 + W(U)\, dx, \qquad U \in \mathscr{M}_{1,2}^{L,M}. \tag{4.1}$$

It follows from standard arguments that $E_{L,M}$ attains its minimum at some $U_{L,M} \in \mathscr{M}_{1,2}^{L,M}$ and $U_{L,M} \in C^{2,\alpha}(R_{L,M})$ satisfies the equation (1.1) in $R_{L,M}$. Moreover, hypothesis (1.10) and the maximum principle (applied to $|U|^2$) yield an *a priori* bound $\|U\|_{L^\infty} \le C$ with constant C independent of L, M. Elliptic regularity then provides the stronger estimate,

$$\|U\|_{C^{2,\alpha}(\overline{R_{L,M}})} \le C_1 \tag{4.2}$$

for any fixed $\alpha \in (0, 1)$, with $C_1 = C_1(\alpha)$ independent of L, M. Finally, we may obtain an upper bound on the energy of the minimizer,

$$\begin{aligned} E_{L,M}(U_{L,M}) &\le E_{L,M}(\Phi_{L,M}) \\ &\le \int_{R_{L,M}} \frac{1}{2}\left[\left|\frac{\partial \varphi_M^1}{\partial x_1}\right|^2 \left(\frac{x_2 + L}{2L}\right)^2 + \left|\frac{\partial \varphi_M^2}{\partial x_1}\right|^2 \left(\frac{-x_2 + L}{2L}\right)^2\right] \\ &\qquad + W(\Phi_{L,M})\, dx + \int_{-M}^{M} \frac{1}{4L}|\varphi_M^1(x_1) - \varphi_M^2(x_1)|^2\, dx_1 \\ &\le \int_{-L}^{L}\int_{-M}^{M} C \exp\{-2\mu|x_1|\}\, dx_1\, dx_2 + C \\ &\le C_2 L, \qquad \text{for } L > 1, \end{aligned} \tag{4.3}$$

with constant C_2 independent of L, M. (We have used (2.5) in estimating the second line.)

From estimate (4.2) we may pass to the limit as $M \to \infty$ along a subsequence to obtain $U_{L,M} \to U_L$ in $C^{2,\alpha}(K)$ for any compact $K \subset T_L$.

Theorem 4.1 *U_L satisfies (1.1) in T_L, $\|U_L\|_{C^{2,\alpha}(\overline{T_L})} \le C_1$, and*

$$E_L(U_L) := \int_{T_L} \frac{1}{2}|\nabla U_L|^2 + W(U_L)\, dx \le C_2 L \tag{4.4}$$

where C_1, C_2 are as in (4.2), (4.3). Moreover, $U_L(x_1, L) = z_1(x_1)$, $U_L(x_1, -L) = z_2(x_1)$, $\gamma \circ U_L = U_L \circ \gamma$, and

$$U_L(x_1, x_2) \to \mathbf{a} \quad \text{uniformly in } x_2 \text{ as } x_1 \to -\infty. \tag{4.5}$$

Proof. We only need to prove (4.5), as the other assertions follow easily from estimates (4.2) and (4.3). Suppose there were a $\delta > 0$ and a sequence of points $\{x^n = (x_1^n, x_2^n)\} \subset T_L$ such that $x_1^n \to -\infty$ and $|U_L(x^n) - \mathbf{a}| > \delta$. Since $U_L(x_1, L) = z_1(x_1) \to \mathbf{a}$ as $x_1 \to -\infty$, we know that we may choose x^n such that $|U_L(x^n) - \mathbf{c}| > \delta$ as well, where $\mathbf{c}$ is any other zero of W. By estimate (4.2) we may conclude that U_L remains away from the minima of W on a sequence of disks: there exist $\varepsilon, \rho > 0$ such that

$$W(U_L(x)) \ge \varepsilon \quad \text{for all } |x - x^n| \le \rho,\ n = 1, 2, \ldots \tag{4.6}$$

Without loss, we may assume that these disks $B_\rho(x^n)$ are disjoint, and hence

$$E_L(U_L) \ge \sum_{n=1}^{\infty} \int_{B_\rho(x^n)} W(U_L)\, dx = +\infty,$$

which contradicts (4.4).
$\diamondsuit$

We now introduce a variational framework for the problem in T_L in analogy to our renormalized energy $\mathscr{E}$ introduced in the previous section. For

$$U \in \mathscr{M}_{1,2}^L := \left\{ \begin{array}{c} U \in \left(H^1_{loc}(T_L)\right)^2 \cap \left(C^0(T_L)\right)^2 : \\ U \circ \gamma(x) = \gamma \circ U(x), \\ U(x) \to \mathbf{a} \text{ as } x_1 \to -\infty, \text{ uniformly in } x_2, \\ U(x_1, L) = z_1(x_1), \quad U(x_1, -L) = z_2(x_1) \end{array} \right\},$$

define

$$\mathscr{E}_L(U) = \int_{-L}^{L} \left[\int_{-\infty}^{\infty} \frac{1}{2}|\nabla U|^2 + W(U)\, dx_1 - e \right] dx_2. \tag{4.7}$$

Then $U_L \in \mathscr{M}_{1,2}^L$ by Theorem 4.1, and for any $U \in \mathscr{M}_{1,2}^L$, $0 < \mathscr{E}_L(U) = E_L(U) - 2eL$.

Define $\sigma_{1,2}^L := \inf\{\mathscr{E}_L(U) : \ U \in \mathscr{M}_{1,2}^L\}$. Since $\mathscr{M}_{1,2}^L \subset \mathscr{M}_{1,2}$ we clearly have $\sigma_{1,2}^L \ge \sigma_{1,2} > 0$.

Lemma 4.2 $\mathscr{E}_L(U_L) = \sigma_{1,2}^L$.

Proof. Suppose not, and there exists $U \in \mathscr{M}_{1,2}^L$ such that $\mathscr{E}_L(U) < \mathscr{E}_L(U_L)$ (or equivalently, $E_L(U) < E_L(U_L)$.) Let $\alpha = \min\{E_L(U_L) - E_L(U), 1\}$. Then, there exists $M > 0$ sufficiently large such that:

$$E_{L,M}(U_{L,M}) - E_L(U) > \frac{\alpha}{2}, \tag{4.8}$$

$$E_{L,M}(U) - E_{L,M-1}(U) < \frac{\alpha}{8}, \tag{4.9}$$

$$|U(-M+1, x_2) - \mathbf{a}| < \frac{1}{4\sqrt{(1+\lambda)L}}\sqrt{\alpha}, \tag{4.10}$$

$$\int_{-L}^{L} \left|\frac{\partial U}{\partial x_2}(-M+1, x_2)\right|^2 dx_2 < \frac{\alpha}{8}. \tag{4.11}$$

Statements (4.8)–(4.10) follow trivially from the definition of the class $\mathscr{M}_{1,2}^L$. We derive (4.11) from Fubini's Theorem and the fact that

$$\int_{T_L} \left|\frac{\partial U}{\partial x_2}(x_1, x_2)\right|^2 dx < E_L(U) < +\infty.$$

Let

$$\tilde{U} = \begin{cases} U(x), & \text{if } |x_1| \le M-1,\ x \in T_L, \\ U(-M+1, x_2)[x_1+M] + \mathbf{a}[-M+1-x_1], & \text{if } -M \le x_1 \le -M+1, \\ U \circ \gamma = \gamma \circ U, & \text{if } M-1 \le x_1 \le M. \end{cases}$$

Then $\tilde{U} \in \mathscr{M}_{1,2}^L$ and

$$\begin{aligned} &|E_{L,M}(\tilde{U}) - E_{L,M}(U)| \le |E_{L,M}(U) - E_{L,M-1}(U)| \\ &\quad + \int_{-L}^{L}\int_{-M+1}^{-M} \left(|U(-M+1,x_2) - \mathbf{a}|^2 + \left|\frac{\partial U}{\partial x_2}(-M+1,x_2)[x_1+M]\right|^2 \right) \\ &\quad\quad +2\lambda|U(-M+1,x_2) - \mathbf{a}|^2[x_1+M]^2\, dx_1\, dx_2 \\ &\le \frac{\alpha}{4} + (2L + 2\lambda L)\frac{1}{16(1+\lambda)L}\alpha \le \frac{\alpha}{2}. \end{aligned}$$

In conclusion,

$$E_{L,M}(\tilde{U}) \le E_{L,M}(U) + \frac{\alpha}{2} < E_{L,M}(U_{L,M}),$$

which contradicts the fact that $U_{L,M}$ minimizes $E_{L,M}$ in $\mathscr{M}_{1,2}^{L,M}$. This concludes the proof of Lemma 4.2.
♢

Since U_L may be trivially extended to $\mathbb{R}^2$ as an element of $\mathscr{M}_{1,2}$, (and at the same time an element of $\mathscr{M}_{1,2}^{L'}$ for all $L' > L$,) we may immediately conclude that $\sigma_{1,2}^L$ is decreasing in L and $\sigma_{1,2}^L \ge \sigma_{1,2}$ for all L.

Lemma 4.3

$$\lim_{L\to\infty} \sigma_{1,2}^L = \sigma_{1,2}.$$

Proof. We show that if $U \in \mathscr{M}_{1,2}$ is chosen close to the infimum of $\mathscr{E}$, then we can construct a test function $V \in \mathscr{M}_{1,2}^L$ for L large with essentially the same energy $\mathscr{E}_L$.

To this end, given any $\varepsilon > 0$ there exists $U \in \mathscr{M}_{1,2}$ such that $\sigma_{1,2} \leq \mathscr{E}(U) < \sigma_{1,2}+\frac{\varepsilon}{2}$. In particular, since the integrand $F(U(\cdot, x_2))-e \geq 0$ there exist sequences $L_n^{\pm} \to \pm\infty$ such that $F(U(\cdot, L_n^{\pm})) - e \to 0$. Set $v_n^{\pm}(x_1) = U(x_1, L_n^{\pm}) \in \mathscr{S}_{ab}$. By Lemma 2.4 we have:

$$v_n^+ \to z_1, \quad v_n^- \to z_2 \quad \text{in } H^1 \cap L^\infty(\mathbb{R}).$$

Choose n sufficiently large such that:

$$\sup_{x_1\in\mathbb{R}} |v_n^+ - z_1| \leq 1, \quad \sup_{x_1\in\mathbb{R}} |v_n^- - z_2| \leq 1, \tag{4.12}$$

$$\left| \int_{-\infty}^{\infty} \left(|v_n^+|^2 - |z_1|^2 \right) dx_1 \right| < \frac{\varepsilon}{2} \tag{4.13}$$

$$\left| \int_{-\infty}^{\infty} \left(|v_n^-|^2 - |z_2|^2 \right) dx_1 \right| < \frac{\varepsilon}{2} \tag{4.14}$$

$$\|v_n^+ - z_1\|_{L^2(\mathbb{R})} < \frac{\varepsilon}{4\beta_0}, \quad \|v_n^- - z_2\|_{L^2(\mathbb{R})} < \frac{\varepsilon}{4\beta_0}, \tag{4.15}$$

where we denote

$$\beta_0 := \|\nabla W(z_1)\|_2, + \|\nabla W(z_2)\|_2 + \beta_1 + 1,$$
$$\beta_1 := \sup\{|D^2 W(\xi)| : \xi \in \mathbb{R}^2,\ |\xi| \leq \|z_1\|_\infty + \|z_2\|_\infty + 1\}.$$

By translating U in the x_2-coordinate we may assume that $L_n^+ = -L_n^- := L$.

Now define $V \in \mathscr{M}_{1,2}^{L+1}$ via:

$$V(x) = \begin{cases} z_1(x_1), & \text{if } x_2 > L+1; \\ z_1(x_1)[x_2 - L] + v_n^+(x_1)[L+1-x_2], & \text{if } L < x_2 \leq L+1; \\ U(x_1, x_2), & \text{if } -L \leq x_2 \leq L; \\ z_2(x_1)[-x_2 - L] + v_n^-(x_1)[L+1+x_2], & \text{if } -L-1 \leq x_2 < -L; \\ z_2(x_1), & \text{if } x_2 < -L-1. \end{cases}$$

We will compare the value of $\mathscr{E}_{L+1}(V) = \mathscr{E}(V)$ with that of $\mathscr{E}_L(U)$. First, we estimate from above:

$$\begin{aligned} I_1 &:= \frac{1}{2} \int_{-\infty}^{\infty} \int_L^{L+1} \left|z_1'(x_1)[x_2 - L] + v_n^+(x_1)[L+1-x_2]\right|^2 - |z_1'(x_1)|^2 \, dx_2\, dx_1 \\ &\leq \frac{1}{2} \int_L^{L+1} \int_{-\infty}^{\infty} [L+1-x_2] \left(|v_n^+(x_1)|^2 - |z_1'(x_1)|^2\right) dx_1\, dx_2 \leq \frac{\varepsilon}{8}, \end{aligned} \tag{4.16}$$

where we have used convexity and (4.13). From (4.16) we immediately have:

$$I_2 := \frac{1}{2}\int_{-\infty}^{\infty}\int_{L}^{L+1} |z_1(x_1) - v_n^+(x_1)|^2\, dx_2\, dx_1 < \frac{\varepsilon}{16}.$$

Expanding in Taylor's series,

$$\begin{aligned} I_3 &:= \int_{-\infty}^{\infty}\int_{L}^{L+1} W(V) - W(z_1)\, dx_2\, dx_1 \\ &\le \int_{-\infty}^{\infty}\int_{L}^{L+1} \left(|\nabla W(z_1)|\,|V - z_1| + \beta_1 |V - z_1|^2\right)\, dx_2\, dx_1 \\ &\le \|\nabla W(z_1)\|_2 \|v_n^+ - z_1\|_2 + \beta_1 \|v_n^+ - z_1\|_2^2 \le \frac{\varepsilon}{4}. \end{aligned}$$

Taking into account that

$$I_4 := \int_{L}^{L+1}\left(\int_{-\infty}^{\infty} \frac{1}{2}|z_1'(x_1)|^2 + W(z_1)\, dx_1 - e\right)\, dx_2 = 0,$$

we have:

$$\int_{L}^{L+1} [F(V(\cdot, x_2)) - e]\, dx_2 = I_1 + I_2 + I_3 + I_4 < \frac{\varepsilon}{2}.$$

Analogous computations over $x_2 \in [-L-1, -L]$ together with the fact that $V = U$ for $x_2 \in [-L, L]$ then yield $\mathscr{E}_{L+1}(V) - \mathscr{E}(U) \le \mathscr{E}_{L+1}(V) - \mathscr{E}_L(U) < \frac{\varepsilon}{2}$, and hence we have:

$$\sigma_{1,2}^{L+1} \le \mathscr{E}_{L+1}(V) \le \mathscr{E}(U) + \frac{\varepsilon}{2} \le \sigma_{1,2} + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the lemma is proven.
♢

5 Passing to the limit

Before presenting the proof of Theorem 3.3, we address the important role of the symmetry assumption $U \circ \gamma = \gamma \circ U$ imposed on solutions (and on the potential W). In particular, the symmetry allows us to fix a horizontal center for the strip solutions, which will allow us to show that the limit function satisfies the asymptotic conditions (1.4) and (1.5). Here we drop the symmetry assumption and give an example of a minimizing sequence in which the "sliding" of an interior layer leads us to a trivial limit. We let U_L be the minimizer of $\mathscr{E}_L$ in $\mathscr{M}_{1,2}^L$, as derived in the previous section, and set

$$V_L = \begin{cases} z_1(x_1), & \text{if } x_2 \ge L + L^3; \\ z_1\left(x_1 - \frac{L+L^3-x_2}{L^2}\right), & \text{if } L \le x_2 \le L + L^3; \\ U_L(x_1 - L, x_2), & \text{if } |x_2| \le L; \\ z_2\left(x_1 - \frac{L+L^3+x_2}{L^2}\right), & \text{if } -L - L^3 \le x_2 \le -L; \\ z_2(x_1), & \text{if } x_2 < -L - L^3. \end{cases}$$

Then, $V_L \in [H^1_{loc} \cap C^0(\mathbb{R}^2)]^2$, and satisfies the desired asymptotic conditions (1.5)–(1.7). (Except for the fact that it is not symmetric, $\gamma \circ V_L \neq V_L \circ \gamma$, V_L would belong to $\mathscr{M}_{1,2}$.) Moreover, an easy computation coupled with Lemma 4.3 shows that $\mathscr{E}(V_L) \to \sigma_{1,2}$ as $L \to \infty$. Nevertheless, $V_L \to \mathbf{a}$ uniformly on compact subsets of $\mathbb{R}$.

Note that such minimizing sequences can still be "normalized" by a horizontal translation, and $\tilde{V}_L(x) = V_L(x_1 + L, x_2)$ *will* converge to a nontrivial solution satisfying the correct asymptotic conditions. We conjecture that the above example gives the only possible way in which the sequence U_L can fail to converge, and hence horizontal translation of the U_L will yield a solution with heteroclinic conditions as $|x| \to \infty$. However, this remains an open question.

We may now prove Theorem 3.3. First, by the symmetry of z_1, we have

$$\gamma \circ z_1(0) = z_1 \circ \gamma(0) = z_1(0).$$

Writing $z_i = (z_{i,1}, z_{i,2})$, $i = 1, 2$ in components, we conclude that the first component, $z_{i,1} = 0$, $i = 1, 2$. Furthermore, since the curves do not intersect, (see Proposition 2.3,) we also know that $z_{1,2}(0) \neq z_{2,2}(0)$. Hence we may choose $\xi \in \mathbb{R}$ so that $\xi \in (z_{1,2}(0), z_{2,2}(0))$ but the point $(0, \xi)$ does not lie on any of the trajectories z_α, $\alpha = 1, \ldots, k$. Then, for any L, there exists $t_L \in \mathbb{R}$ with $|t_L| < L$ and

$$U_L(0, t_L) = (0, \xi). \tag{5.1}$$

Next, we define

$$V_L(x_1, x_2) := U_L(x_1, x_2 + t_L), \qquad x \in T_L - t_l,$$

where $T_L - t_L = \{x \in \mathbb{R}^2 : \ (x_1, x_2 + t_L) \in T_L\}$. By the local uniform Schauder estimate (see (4.2)), we may extract a sequence $L_n \to \infty$ such that

$$V_{L_n}(x) \to V(x) \quad \text{in } C^{2,\alpha}(K),$$

for any compact set $K \subset \mathscr{R}$, where the set $\mathscr{R}$ is the limit $\mathscr{R} = \lim_{n\to\infty}[T_{L_n} - t_{L_n}]$. Clearly V satisfies (1.1) in $\mathscr{R}$, and

$$V(0) = (0, \xi). \tag{5.2}$$

We now consider the three cases which are possible: (1) $\mathscr{R} = \mathbb{R}^2$; (2) $\mathscr{R} = \mathbb{R}^{2+} = \{x : \ x_2 \geq -\kappa\}$, with some κ finite; (3) $\mathscr{R} = \mathbb{R}^{2-} = \{x : \ x_2 \leq \kappa\}$, with some κ finite.

5.1 Case (1), $\mathscr{R} = \mathbb{R}^2$

Note first that for any $M > 0$ fixed, $T_M \subset T_{L_n} - t_{L_n}$ for all n large enough. Writing $L = L_n$, we have:

$$(5.3)\quad E_M(V_L) \le E_L(U_L) - \int_{M+t_L}^{L} F(U_L(\cdot,x_2)\,dx_2 - \int_{-L}^{-M+t_L} F(U_L(\cdot,x_2)\,dx_2$$
$$\le \sigma_{1,2}^L + 2eL - 2e(L-M) = \sigma_{1,2}^L + 2eM\,.$$

Letting $L = L_n \to \infty$,

$$(5.4)\qquad E_M(V) \le \sigma_{1,2} + 2eM\,,$$

and letting $M \to \infty$ we have

$$(5.5)\qquad \mathscr{E}(V) \le \sigma_{1,2}.$$

It remains to prove that $V \in \mathscr{M}_{1,2}$.

Lemma 5.1 *$V(x_1,x_2) \to$ **a** as $x_1 \to -\infty$ and $V(x_1,x_2) \to$ **b** as $x_1 \to +\infty$, uniformly in x_2.*

Proof. First, we claim that $W(V(x_1,x_2)) \to 0$ as $|x_1| \to \infty$ for each fixed x_2. Indeed, if this were not the case, there would exist $\delta > 0$ and a sequence of points $x^n = (x_1^n, x_2^0)$, with $|x_1^n| \to \infty$ and x_2^0 fixed, such that $W(V(x^n)) > \delta$. Since V satisfies the uniform Schauder estimate (4.2) we may in fact conclude that $W(V(x)) > \delta/2$ for all $x \in B_\rho(x_n)$, $n = 1,2,\ldots$, for some fixed $\rho > 0$. From (5.4) we then obtain the contradiction:

$$\sigma_{1,2} + 2eM \ge E_M(V) \ge \sum_{n=1}^{\infty} \int_{B_\rho(x^n)} W(V(x)) = +\infty.$$

Consequently, the values $V(x_1,x_2)$ accumulate at the (finite set of) zeros of W as $x_1 \to -\infty$. Consider two sequences $x^n = (x_1^n, x_2^0)$, $y^n = (y_1^n, y_2^0)$, with $x_1^n, y_1^n \to -\infty$ and x_2^0, y_2^0 fixed, and $V(x^n) \to \mathbf{c}_1$, $V(y^n) \to \mathbf{c}_2$, $W(\mathbf{c}_1) = 0 = W(\mathbf{c}_2)$. Suppose $\mathbf{c}_1 \ne \mathbf{c}_2$. By the uniform continuity of V and hypothesis (1.9), on each line segment connecting the points x^n and y^n there must exists a point $\xi^n = (\xi_1^n, \xi_2^n)$ with $W(V(\xi_n)) > \delta$ for some fixed $\delta > 0$. Clearly $\xi_1^n \to -\infty$, and the argument of the above paragraph again leads to a contradiction of the energy bound (5.4). Hence, $\mathbf{c}_1 = \mathbf{c}_2$ and the limit set $V(x_1,x_2)$ as $x_1 \to -\infty$ is *unique*: there exists $\mathbf{c}$ with $W(\mathbf{c}) = 0$ such that $V(x_1,x_2) \to \mathbf{c}$ as $x_1 \to -\infty$ for all fixed x_2.

Next, we show that necessarily $\mathbf{c} = \mathbf{a}$. By hypothesis (1.14) there exists $\beta_0 > 0$ such that

$$(5.6)\qquad e_{ab} + \beta_0 \le e_{ac} + e_{cb}$$

holds for every zero $\mathbf{c}$ of W different from $\mathbf{a}$ or $\mathbf{b}$. Fix $\delta < \frac{\beta_0}{6C_0}$ with C_0 as in Lemma 2.1. By a diagonal procedure, there exists a sequence $x^n = (-x_1^n, n)$ with $x_1^n \to +\infty$ and $|V(x^n) - \mathbf{c}| < \delta/4$. Using the Schauder estimate (4.2) there exists $\rho > 0$ so that:

$$|V(x) - \mathbf{c}| < \frac{\delta}{2}, \qquad x \in \cup_{n=1}^{\infty} Q_\rho(x^n),$$

with cubes $Q_\rho(x^n) = [-x_1^n - \rho, -x_1^n + \rho] \times [n - \rho, n + \rho]$. Choose an integer N with

$$N > 3\sigma^1_{1,2}/\beta_0\rho, \quad (\sigma^1_{1,2} = \sigma^L_{1,2},\ L = 1,)$$

and fix a compact set K with $\cup_{n=1}^N Q_\rho(x^n) \subset K$. For $L > 1$ sufficiently large we have $K \subset T_L$ and (by the uniform convergence $V_L \to V$ on compact sets)

$$|V_L(x) - \mathbf{c}| < \delta, \qquad x \in \cup_{n=1}^N Q_\rho(x^n). \tag{5.7}$$

By Theorem 4.1, $V_L(x_1, x_2) \to \mathbf{a}$ as $x_1 \to -\infty$ uniformly in x_2, so there exists $R > 0$ such that

$$|V_L(x) - \mathbf{a}| < \delta \qquad \text{for all } x_1 < -R,$$

and by symmetry

$$|V_L(x) - \mathbf{b}| < \delta \qquad \text{for all } x_1 > R.$$

Take first the case where $\mathbf{c} \neq \mathbf{b}$. Applying Lemma 2.1 to the intervals $[-R, -x_1^n]$ and $[-x_1^n, R]$ we obtain the following lower bounds holding in horizontal strips, $x_2 \in \cup_{n=1}^N [n-\rho, n+\rho]$:

$$\int_{-R}^{-x_1^n} \left|\frac{\partial V_L}{\partial x_1}\right|^2 + W(V_L(x_1, x_2))\, dx_1 \ \geq\ e_{ac} - C_0\delta \geq e_{ac} - \frac{1}{6}\beta_0,$$
$$\int_{-x_1^n}^{+R} \left|\frac{\partial V_L}{\partial x_1}\right|^2 + W(V_L(x_1, x_2))\, dx_1 \ \geq\ e_{cb} - C_0\delta \geq e_{ac} - \frac{1}{6}\beta_0.$$

In particular, this implies the following contradiction,

$$\begin{aligned}
\sigma^1_{1,2} &\geq \sigma^L_{1,2} \geq \int_{-L}^{L} \left(\int_{-\infty}^{\infty} \left|\frac{\partial V_L}{\partial x_1}\right|^2 + W(V_L(x_1, x_2))\, dx_1 \right) - e_{ab}\, dx_2 \\
&\geq \sum_{n=1}^{N} \int_{n-\rho}^{n+\rho} \left(\int_{-R}^{R} \left|\frac{\partial V_L}{\partial x_1}\right|^2 + W(V_L(x_1, x_2))\, dx_1 \right) - e_{ab}\, dx_2 \\
&\geq 2\rho N \left[e_{ac} + e_{cb} - e_{ab} - \frac{1}{3}\beta_0 \right] \geq \frac{4}{3} N\beta_0 > 4\sigma^1_{1,2},
\end{aligned}$$

by the choice of N. Hence, the limit value of V as $x_1 \to -\infty$ must be either $\mathbf{a}$ or $\mathbf{b}$. If the limit were actually $\mathbf{b}$, we note that when (5.7) holds with $\mathbf{c} = \mathbf{b}$, then we have its reflection,

$$|V_L(x) - \mathbf{a}| < \delta, \qquad x \in \cup_{n=1}^N Q_\rho(-x^n).$$

In other words, in each strip $x_2 \in [n-\rho, n+\rho]$ we observe *three* transitions in V_L, from $\mathbf{a}$ to $\mathbf{b}$ in $[-R, -x_1^n]$ and $[x_1^n, R]$ and from $\mathbf{b}$ to $\mathbf{a}$ in $[-x_1^n, x_1^n]$. Estimating the energy as above we obtain:

$$\begin{aligned}
\sigma^1_{1,2} &\geq \sigma^L_{1,2} \geq \sum_{n=1}^{N} \int_{n-\rho}^{n+\rho} \left(\int_{-R}^{R} \left|\frac{\partial V_L}{\partial x_1}\right|^2 + W(V_L(x_1, x_2))\, dx_1 - e_{ab} \right) dx_2 \\
&\geq 2\rho N (3e_{ab} - \frac{1}{2}\beta_0) \geq 2N\beta_0 > 6\sigma^1_{1,2},
\end{aligned}$$

again a contradiction. In conclusion, $V \to \mathbf{a}$ as $x_1 \to -\infty$.

By symmetry we also have $V \to \mathbf{b}$ as $x_1 \to +\infty$, and hence $V(\cdot, x_2) \in \mathscr{S}_{ab}$ for each x_2.

Finally, we show that this convergence is uniform in x_2. Suppose not: by familiar arguments there would exist constants $\delta, \rho > 0$ and a sequence $x^n = (x_1^n, x_2^n)$ with $x_1^n \to -\infty$ and $|x_2^n| \to +\infty$ such that

$$|V(x) - \mathbf{a}| > \delta \qquad \text{for all } x \in Q_\rho(x^n).$$

Let $R_0 > 0$ be defined by:

$$|z(x_1) - \mathbf{a}| < \frac{\delta}{2} \qquad \text{for all } x_1 < -R_0 \text{ and } z \in \mathscr{Z}.$$

Then, if n is large and $x_2 \in [x_2^n - \rho, x_2^n + \rho]$ we have

$$\|z(\cdot) - V(\cdot, x_2)\|_{L^\infty(\mathbb{R})} > \frac{\delta}{2}.$$

By Lemma 2.4 there exists $\varepsilon > 0$ such that $F(V(\cdot, x_2)) \geq e_{ab} + \varepsilon$ for each $x_2 \in [x_2^n - \rho, x_2^n + \rho]$, n sufficiently large. However this contradicts the energy bound (5.5), and hence the proof is complete.
$\diamondsuit$

Next we show that V attains the correct limits as $x_2 \to \pm\infty$. For $y \in \mathbb{R}$, let $V^y = V(x_1, x_2 + y)$. Since V^y satisfies the same uniform Schauder estimate as V, we may extract subsequences $y_n^\pm \to \pm\infty$ with $V^{y_n^\pm} \to v^\pm$ in $C^{2,\alpha}(K)$ for every compact K.

Lemma 5.2 *$v^\pm$ does not depend on x_2, $v^\pm = v^\pm(x_1)$.*

Proof. For each fixed x_2, $V(x_1, x_2)$ gives an admissible function of class $\mathscr{S}_{ab}$ in x_1, and hence:

$$\int_{-M}^{M} F(V(\cdot, x_2))\, dx_2 \geq 2eM. \tag{5.8}$$

From (5.4),

$$\sigma_{1,2} + 2eM \geq E_M(V) = \int_{-M}^{M}\int_{-\infty}^{+\infty} \left|\frac{\partial V}{\partial x_2}\right|^2 dx_1\, dx_2 + \int_{-M}^{M} F(V(\cdot, x_2))\, dx_2,$$

and hence by applying (5.8) and passing to the limit $M \to \infty$ we obtain:

$$\int_{\mathbb{R}^2} \left|\frac{\partial V}{\partial x_2}\right|^2 dx_1\, dx_2 \leq \sigma_{1,2}. \tag{5.9}$$

We now argue as in Lemma 4.5 of [3] to conclude that

$$\sup_{x_1} \left|\frac{\partial V}{\partial x_2}\right|(x_1, x_2) \to 0 \qquad \text{as } |x_2| \to +\infty. \tag{5.10}$$

Indeed, suppose instead that there exists a sequence of points $x_n = (x_{n,1}, x_{n,2}) \in \mathbb{R}^2$ with $|x_{n,2}| \to +\infty$ and an $\eta > 0$ such that $|V_{x_2}|(x_{n,1}, x_{n,2}) \geq \eta > 0$. Invoking again the uniform $C^{2,\alpha}$ estimate satisfied by V, we can actually conclude that there exists $r > 0$ (independent of n) so that

$$\left|\frac{\partial V}{\partial x_2}\right|(x) \geq \frac{\eta}{2} \quad \text{for all } |x - x_n| < r.$$

But clearly this contradicts the square integrability of $\frac{\partial V}{\partial x_2}$, (5.9), and hence (5.10) must hold.

Finally, if we fix x and write

$$\frac{\partial v^{\pm}}{\partial x_2}(x) = \left[\frac{\partial v^{\pm}}{\partial x_2}(x_1, x_2) - \frac{\partial V^{y_n^{\pm}}}{\partial x_2}(x_1, x_2)\right] + \frac{\partial V}{\partial x_2}(x_1, x_2 + y_n^{\pm}),$$

as $n \to \infty$ the first term tends to zero uniformly on compact sets, while the second vanishes by (5.10), and we obtain $\frac{\partial v^{\pm}}{\partial x_2}(x) = 0$.
♢

We observe that, *a priori*, $V(x_1, x_2)$ could have different subsequential limits as $x_2 \to \pm\infty$. This will not be the case, however, as we will show later on that V must tend to the given functions z_1, z_2 as $x_2 \to \pm\infty$.

By Lemma 5.1, $v^{\pm} \in \mathscr{S}_{ab}$. In fact they must be *minimizers*:

Lemma 5.3 *$v^{\pm}$ achieve $\min_{v \in \mathscr{S}_{ab}} F(v)$. (i.e., $v^{\pm} \in \mathscr{Z}$.)*

Proof. Suppose the contrary; then there exist sequences $y_n^{\pm} \to \pm\infty$ such that

$$\sup_{x_1 \in \mathbb{R}} |V(x_1, y_n^{\pm}) - z(x_1)| > 2\delta_0$$

for some fixed $\delta_0 > 0$ and for all elements $z \in \mathscr{Z}$. By the uniform Schauder estimate satisfied by V we may in fact obtain a constant radius $r > 0$ (independent of n) such that

$$\sup_{x_1 \in \mathbb{R}} |V(x_1, x_2) - z(x_1)| > \delta_0 \quad \text{for all } x_2 \in \mathbb{R} \text{ with } |x_2 - y_n^{\pm}| < r.$$

By Lemma 5.3, we conclude that $F(V(\cdot, y_n^{\pm})) > e + \varepsilon_0$ for some $\varepsilon_0 > 0$ independent of n. From this lower bound we would then conclude that $\mathscr{E}(V) = +\infty$, which contradicts (5.5).
♢

We next show that minimality forces the limit V to tend to the "correct" curves $|x_2| \to \infty$, $v^+ = z_1$ and $v^- = z_2$. First we eliminate the possibility that $v^+ = z_2$. Combining the fact that $V(0) = (0, \xi)$ lies away from any of the trajectories in $\mathscr{Z}$, the uniform $C^{2,\alpha}$ estimate on V, and Lemma 2.4, we conclude that there exist $\delta, \rho > 0$ such that

$$F(V_L(\cdot, x_2)) > e + \delta \quad \text{for all } |x_2| \leq \rho. \tag{5.11}$$

Now, if indeed $v^+ = z_2$, then (by the local uniform convergence of $V(x_1, x_2 + y_n)$ as the subsequence $y_n \to \infty$) for any $\varepsilon > 0$ and $M > 0$ there exists $y_0 > M + 2\rho$ such that

$$|V(x_1, x_2 + y_0) - z_2(x_1)| < \frac{\varepsilon}{2} \quad \text{for all } x \in [-M, M]^2. \tag{5.12}$$

Hence, if L_0 is chosen sufficiently large, then we also have

$$|V_L(x_1, x_2 + y_0) - z_2(x_1)| < \varepsilon \quad \text{for all } x \in [-M, M]^2 \text{ and } L > L_0.$$

By the definition of V_L we have

$$\int_{-M+y_0}^{M+y_0} (F(V_L(\cdot, x_2)) - e)\, dx_2 < \sigma_{1,2}^L < \sigma_{1,2}^1.$$

In particular, there exists $y_1 \in [-M + y_0, M + y_0]$ such that

$$0 < F(V_L(\cdot, y_1)) - e < \frac{\sigma_{1,2}^1}{2M}. \tag{5.13}$$

By fixing M sufficiently large, we then have:

$$F(V_L(\cdot, y_1)) - e < \frac{\varepsilon}{8} \quad \text{and} \quad \|V_L(\cdot, y_1) - z_2\|_{H^1(\mathbb{R})} < \frac{\varepsilon}{8}. \tag{5.14}$$

Define a variant of V_L in $[-L - t_L, L - t_L]$ by:

$$\tilde{V}_L(x) := \begin{cases} V_L(x), & \text{if } y_1 \le x_2 \le L - t_L; \\ V_L(x_1, y_1)[x_2 - y_1 + 1] + z_2(x_1)[y_1 - x_2], & \text{if } y_1 - 1 < x_2 < y_1; \\ z_2(x_1), & \text{if } -L - t_L \le x_2 \le y_1 - 1, \end{cases}$$

and $\tilde{U}_L(x) = \tilde{V}_L(x_1, x_2 - t_L) \in \mathscr{M}_{1,2}^L$. We will now show that the energy of $\tilde{U}_L$ has been reduced below the value of $\sigma_{1,2}^L$. Proceeding as in (4.16), we may fix $\varepsilon > 0$ sufficiently small such that:

$$I := \int_{y_1 - 1}^{y_1} \left(F(\tilde{V}_L(\cdot, x_2)) - e\right) dx_2 < \rho\delta. \tag{5.15}$$

Breaking the integral into its parts and applying (5.11) we obtain:

$$\begin{aligned} \mathscr{E}_L(\tilde{U}_L) &\le \mathscr{E}_L(U_L) - \int_{-L-t_L}^{y_1 - 1} (F(V_L(\cdot, x_2)) - e)\, dx_2 + I \\ &\le \sigma_{1,2}^L - \int_{-\rho}^{\rho} (F(V_L(\cdot, x_2)) - e)\, dx_2 + \rho\delta \\ &\le \sigma_{1,2}^L - 2\rho\delta + \rho\delta < \sigma_{1,2}^L, \end{aligned}$$

which contradicts the definition of $\sigma_{1,2}^L$ as the infimum of $\mathscr{E}_L$ over $\mathscr{M}_{1,2}^L$. In conclusion, we cannot have $v^+ = z_2$. By similar arguments, we may also conclude that $v^- \ne z_1$.

Finally, we must eliminate the possibility that $v^{\pm} = z_\alpha$ for some $\alpha \neq i, j$. Exactly as in deriving (5.14) above, we may conclude that for any $\eta > 0$ there exist values $y_2 > y_1 + 2$ such that

$$0 \le F(V_L(\cdot, y_m)) - e < \eta, \quad \|V_L(\cdot, y_m) - z_\alpha\|_{L^\infty \cap H^1(\mathbb{R})} < \eta, \; m = 1, 2. \tag{5.16}$$

Now we split V_L into two functions, one connecting z_1 to z_α and the other connecting z_α to z_2: Define $\hat{V}_{L,2}$ on the strip $\mathbb{R} \times [y_2 - 1, L - t_L]$ by:

$$\hat{V}_{L,2}(x) := \begin{cases} V_L(x), & \text{if } y_2 \le x_2 \le L - t_L; \\ V_L(x_1, y_2)[x_2 - y_2 + 1] + z_\alpha(x_1)[y_2 - x_2], & \text{if } y_2 - 1 < x_2 < y_2, \end{cases}$$

and $\hat{V}_{L,1}$ on $\mathbb{R} \times [-L - t_L, y_1 + 1]$ by:

$$\hat{V}_{L,1} := \begin{cases} V_L(x_1, y_1)[1 + y_1 - x_1] + z_\alpha(x_1)[x_1 - y_1], & \text{if } y_1 < x_1 < y_1 + 1; \\ V_L(x_1, x_2) & \text{if } -L - t_L \le x_2 \le y_1 - 1. \end{cases}$$

By shifting in the x_2-direction, we set $\hat{U}_{L,m}(x) = \hat{V}_{L,m}(x_1, x_2 - T_{L,m})$, $m = 1, 2$, where $T_{L,m}$ are chosen such that $U_{L,m}$ are now defined on symmetric strips, $\mathbb{R} \times [-L_m, L_m]$, $m = 1, 2$.

Following the same reasoning as in (4.16) and (5.15), given any $\varepsilon > 0$ we may choose $\eta > 0$ sufficiently small such that:

$$I_1 := \int_{y_1}^{y_1+1} \left(F(\hat{V}_{L,1}(\cdot, x_2)) - e\right) \, dx_2 < \frac{\varepsilon}{2},$$
$$I_2 := \int_{y_2-1}^{y_2} \left(F(\hat{V}_{L,2}(\cdot, x_2)) - e\right) \, dx_2 < \frac{\varepsilon}{2}.$$

We may then estimate $\sigma_{1,2}^L$ from below:

$$\begin{aligned} \sigma_{1,2}^L = \mathscr{E}_L(U_L) \; &> \; \int_{-L-t_L}^{y_1} \left[\int_{-\infty}^{\infty} \frac{1}{2}|\nabla V_{L,1}|^2 + W(V_{L,1})\, dx_1 - e\right] dx_2 \\ &\quad + \int_{y_2}^{L-t_L} \left[\int_{-\infty}^{\infty} \frac{1}{2}|\nabla V_{L,2}|^2 + W(V_{L,2})\, dx_1 - e\right] dx_2 \\ &= \; \mathscr{E}_{L_1}(\hat{U}_{L_1}) + \mathscr{E}_{L_2}(\hat{U}_{L_2}) - I_1 - I_2 \\ &\ge \; \sigma_{2,\alpha}^{L_1} + \sigma_{\alpha,1}^{L_2} - \varepsilon \ge \sigma_{2,\alpha} + \sigma_{\alpha,1} - \varepsilon. \end{aligned}$$

Noting that $\varepsilon > 0$ is arbitrary, we take the limit $L \to \infty$ in (5.16) to conclude:

$$\sigma_{1,2} \ge \sigma_{2,\alpha} + \sigma_{\alpha,1}. \tag{5.17}$$

The reverse inequality to (5.17) is elementary: Consider the respective minimizers $U_{L,1}$ and $U_{L,2}$ to $\mathscr{E}_L$ in $\mathscr{M}_{2,\alpha}^L$ and $\mathscr{M}_{\alpha,1}^L$ respectively, and glue them together along their common boundary value z_α to form $\bar{U}_L \in \mathscr{M}_{1,2}^{2L}$. Then,

$$\sigma_{1,2} \le \sigma_{1,2}^{2L} \le \mathscr{E}_{2L}(\bar{U}_L) = \mathscr{E}_L(U_{L,1}) + \mathscr{E}_L(U_{L,2}) = \sigma_{2,\alpha}^L + \sigma_{\alpha,1}^L.$$

Passing to the limit $L \to \infty$, we then have

$$\sigma_{1,2} = \sigma_{2,\alpha} + \sigma_{\alpha,1},$$

which contradicts the hypothesis (3.3). In conclusion, $v^+ = z_1$ and identical arguments show that $v^- = z_2$. Together with Lemma 5.1 this implies that $V \in \mathscr{M}_{1,2}$ and we have finished the proof of (3.3) in case (1), $\mathscr{R} = \mathbb{R}^2$.

5.2 *Case (2),* $\mathscr{R} = \mathbb{R}^{2+}$

From the elliptic estimate (4.2) on the strip solutions U_L, we may conclude that there exists V^+ with $V_L \to V^+$ in $C^{2,\alpha}(K)$ for any compact $K \subset \mathbb{R}^{2+}$. Moreover, on the boundary of $\mathbb{R}^{2+}$, $V^+(x_1, -\kappa) = z_2(x_1)$. Repeating the steps (5.3), (5.4) and (5.5) we obtain:

$$\int_{-\kappa}^{\infty} \left[\int_{-\infty}^{\infty} \frac{1}{2} |\nabla V^+|^2 + W(V^+)\, dx_1 - e \right] dx_2 \leq \sigma_{1,2}.$$

Define

$$V(x_1, x_2) := \begin{cases} V^+(x_1, x_2), & \text{if } x_2 \geq -\kappa; \\ z_2(x_1), & \text{if } x_2 \leq -\kappa. \end{cases}$$

Arguing exactly as in case (1) we obtain $V \in \mathscr{M}_{1,2}$, so V attains the minimum $\sigma_{1,2} = \inf_{\mathscr{M}_{1,2}} \mathscr{E}$ and by Proposition 3.2 V is a smooth solution of the equation (1.1) in $\mathbb{R}$. However, if this were the case then $V_{x_2} = \frac{\partial V}{\partial x_2}$ would satisfy the (linear) system,

$$\begin{cases} -\Delta V_{x_2} + D^2 W(V)\, V_{x_2} = 0, & x \in \mathbb{R}^2, \\ V_{x_2}(x_1, x_2) = 0, & x_2 \leq -\kappa. \end{cases}$$

The unique continuation result of N. Garofalo & F. H. Lin, Theorem 4.2 of [9], then implies that $V_{x_2} \equiv 0$ in $\mathbb{R}$, and hence case (2) cannot occur. The same argument eliminates also case (3), and therefore the proof of Theorem 3.3 is complete.
$\diamondsuit$

6 Saddle solutions

By a saddle solution we mean a solution $U(x_1, x_2)$ of (1.1) in $\mathbb{R}^2$ with asymptotic conditions as $|x| \to \infty$,

$$\lim_{x_2 \to +\infty} U(x_1, x_2) = z(x_1), \tag{6.1}$$

$$\lim_{x_2 \to -\infty} U(x_1, x_2) = z(-x_1), \tag{6.2}$$

$$\lim_{x_1 \to +\infty} U(x_1, x_2) = z(x_2), \tag{6.3}$$

$$\lim_{x_1 \to -\infty} U(x_1, x_2) = z(-x_2), \tag{6.4}$$

where z is a (one-dimensional) heteroclinic solution of (1.1). We make the same hypotheses (1.8)–(1.11) on the potential W as for heteroclinic solutions, but now require that no zero of W may lie on the axis of symmetry:

$$W(0,\xi_2) \neq 0 \quad \text{for all } \xi_2 \in \mathbb{R}. \tag{6.5}$$

For any two zeros $\mathbf{p},\mathbf{q}$ of W, we recall that e_{pq} denotes the minimum energy F of a heteroclinic trajectory connecting the wells $\mathbf{p}$ and $\mathbf{q}$, and $\mathscr{Z}_{pq}$ the collection of minimizing paths (normalized by symmetry, $z(-t) = \gamma \circ z(t)$) which connect $\mathbf{p}$ to $\mathbf{q}$. Let

$$e := \min\{e_{pq} : \ \mathbf{p},\mathbf{q} \text{ zeros of } W \text{ with } \mathbf{q} = -\mathbf{p}.\}$$

We prove the following:

Theorem 6.1 *Suppose* $\mathbf{a}$, $\mathbf{b} = -\mathbf{a}$ *are zeros of* W *with* $e_{ab} = e$, *and that* $\mathscr{Z}_{ab}$ *has only finitely many elements. Then there exists a solution of (1.1) satisfying the asymptotic saddle conditions (6.1)–(6.4) for some* $z \in \mathscr{Z}_{ab}$.

As the proof of Theorem 6.1 is very similar to the proofs of Theorem 3.3 above and Theorem 1.1 of [3] we only provide a brief sketch of the essential elements, and leave certain details to the interested reader.

As for the heteroclinic solutions studied earlier we will impose symmetries on our admissible functions in order to avoid the losses of compactness due to translation invariance. First, define the following symmetries of the square:

$$\gamma(\xi_1,\xi_2) = (\xi_1,-\xi_2), \quad \gamma'(\xi_1,\xi_2) = (-\xi_1,\xi_2), \quad \gamma''(\xi_1,\xi_2) = (\xi_2,\xi_1)$$

Then, define the class of admissible functions

$$\tilde{\mathscr{M}} = \left\{ \begin{array}{c} U \in \left(H^1_{loc}(\mathbb{R}^2)\right)^2 \cap \left(C^0(\mathbb{R}^2)\right)^2 : \\ U \circ \gamma(x) = \gamma \circ U(x),\ U \circ \gamma'(x) = \gamma \circ U(x),\ U \circ \gamma''(x) = U(x), \\ \text{and } U \text{ satisfies (6.1)–(6.4) as } |x| \to \infty. \end{array} \right\}$$

The saddle solutions we seek necessarily have infinite energy on $\mathbb{R}^2$, and unlike the heteroclinic solutions (which connect zeros of W on each horizontal slice) we see no obvious way to “renormalize” the energy to make it finite. Nevertheless, we will proceed as in the proof of Theorem 3.3, solving boundary-value problems for (1.1) in squares, and then passing to a limit as the length of the sides tends to infinity. To this end, we denote by $Q_L = [-L,L]^2$ the centered square, and

$$\tilde{\mathscr{M}}_L = \left\{ \begin{array}{c} U \in \left(H^1(Q_L)\right)^2 :\ U \circ \gamma(x) = \gamma \circ U(x),\ U \circ \gamma'(x) = \gamma \circ U(x), \\ U \circ \gamma''(x) = U(x),\ \text{and } U(x_1,L) = z_L(x_1),\ U(L,x_2) = z_L(x_2) \end{array} \right\},$$

where $z_L(t) = z(t)\eta(t/L)$ with smooth cut-off $\eta(t)$ satisfying $\eta(t) = 1$ for $|t| \leq \frac{1}{2}$ and $\eta(t) = 0$ for $|t| \geq 1$. By symmetry, it suffices to do all computations in the upper triangle of Q_L, defined by:

$$T_L = \{(x_1,x_2) :\ 0 \leq x_2 \leq L,\ -x_2 \leq x_1 \leq x_2\}.$$

Consider first the minimization problem in the square Q_L,

$$\min_{u\in\tilde{\mathscr{M}}_L} E_L(u), \quad E_L(u) = \int_{Q_L} \left(\frac{1}{2}|\nabla u|^2 + W(u)\right) dx_1\, dx_2. \tag{6.6}$$

The existence of a minimizer U_L, satisfying equation (1.1) in Q_L and the given boundary conditions on ∂Q_L is standard. In fact, by the uniform *a priori* estimate (4.2) satisfied by all solutions U_L of these boundary-value problems we may extract a convergent subsequence $U_L \to U$ in $C^{2,\alpha}(K)$ for any compact region K. U will solve (1.1) in $\mathbb{R}^2$; the difficulty is to verify the asymptotic conditions (6.1)–(6.4). As in the heteroclinic case, this will be accomplished by means of energy estimates.

We obtain a simple but powerful estimate on the minimum energy $E_L(U_L)$ as follows:

Lemma 6.2 *There exists a constant C_1 independent of L such that $E_L(U_L) \leq 4eL + C_1$.*

This follows from a direct calculation of the energy of a test function φ. Define φ first on $T_L \setminus T_1$ by $\varphi(x) = z_{L=x_2}(x_1)$ if $x_2 > 1$ and $x_1 \in [-x_2, x_2]$. Then we obtain its values on $Q_L \setminus Q_1$ by reflections, $\varphi(\Gamma(x)) = \gamma\varphi(x)$ for $\Gamma = \gamma, \gamma'$, and $\varphi(\gamma''(x)) = \varphi(x)$. Then φ may be extended as a smooth symmetric function inside the square Q_1. The computation is nearly identical to Lemma 4.1 of [3], and hence is omitted here.

In order to compare the solutions U_L for different values of L, we must fix a square Q_M and estimate the energy $E_M(U_L)$ with M fixed. To do this we require a lower bound on the energy as well. For the heteroclinic problem above, we were able to obtain such a lower bound in Proposition 3.1, by using the fact that each horizontal restriction $U(\cdot, x_2)$ is an admissible function for the minimization problem on $\mathbb{R}$, and hence has its one-dimensional energy F bounded below by e. The difference in the saddle geometry is that the horizontal lines making up the triangle T_L now have their endpoints on the diagonals of the square Q_L, and hence we cannot control the boundary values on these (finite) intervals. In order to overcome this geometrical problem we derive a lower bound on the energy based on solutions of Neumann boundary-value problems (see [3].) Define

$$e_{L,N} = \min\left\{\int_{-L}^{L} \left[\frac{1}{2}|z'(t)|^2 + W(z)\right] dt : \ z \in H^1([-L, L]),\ z(-t) = \gamma \circ z(t)\right\}.$$

Standard arguments show that $e_{L,N}$ is attained by one-dimensional solutions of (1.1) on $[-L, L]$ with Neumann boundary conditions on $x = \pm L$. Clearly $e_{L,N} \leq e$, and it follows from Corollary 3.4 of [3] (and the fact that no zero of W lies on the axis of symmetry) that

$$e - e_{L,N} \leq C \exp\{-\nu L\},$$

for positive constants C, ν. From this estimate we may obtain a lower bound on the energy in the annular region $Q_L \setminus Q_M$, $0 \leq M < L$:

$$\begin{aligned}\int_{Q_L\setminus Q_M}\left[\frac{1}{2}|\nabla U|^2+W(U)\right]\,dx \;&\geq\; 4\int_M^L\int_{-x_2}^{x_2}\left[\frac{1}{2}\left|\frac{\partial U}{\partial x_1}\right|^2+W(U)\right]\,dx_1\,dx_2\\ &\geq\; 4\int_L^M\left(e-C\exp\{-\nu x_2\}\right)\,dx_2\\ &\geq\; 4e(L-M)-C_2, \qquad (6.7)\end{aligned}$$

for a constant C_2 independent of L, M.

Thanks to the lower bound (6.7), we may now obtain a sharper upper bound on the minimizers U_L, in terms of the energy E_M with $M\leq L$:

$$E_M(U_L)\leq 4eM+C_1,$$

with constant C_0 independent of L, M, and passing to the limit as $L\to\infty$, we have:

$$E_M(U)\leq 4eM+C_1. \qquad (6.8)$$

From the bound (6.8) we may now argue as in Lemma 5.2 above (or Lemma 4.6 of [3]) to show that $U(x_1,x_2)\to v^+(x_1)$ as $x_2\to+\infty$. Indeed, by the lower bound (6.7) we have

$$\int_{Q_M}\left[\frac{1}{2}\left|\frac{\partial U}{\partial x_1}\right|^2+W(U)\right]\,dx\geq 4eM-C_2.$$

On the other hand, the full energy in Q_M is bounded above, $E_M(U)\leq 4eM+C_1$, by Lemma 6.2. Hence,

$$\int_{Q_M}\left|\frac{\partial U}{\partial x_2}\right|^2\,dx\leq C_3$$

with C_3 independent of M. Passing to the limit as $M\to\infty$,

$$\int_{\mathbb{R}^2}\left|\frac{\partial U}{\partial x_2}\right|^2\,dx\leq C_3<\infty.$$

Now we may follow the steps of Lemma 5.2 to conclude that

$$\lim_{x_2\to+\infty}\sup_{x_1}\left|\frac{\partial U}{\partial x_2}\right|=0.$$

(See also Lemma 4.6 of [3].) Let $U^y(x)=U(x_1,x_2+y)$. Then for any given sequence $\{y_n\}_n$ with $y_n\to\infty$, there is a subsequence, which we still denote by $\{y_n\}_n$, such that U^{y_n} converges uniformly in any compact set in R^2 to a function $v(x_1)$ as $y_n\to\infty$.

Following the argument in the first paragraph of the proof of Lemma 5.1 we may conclude that v must connect zeros of W as $x_1\to\pm\infty$. Since no zero of W lies on the axis of symmetry between **a** and **b**, and since v is itself symmetric, it must join a symmetric pair of minima, say **p** and **q** $=-$**p**. By hypothesis, e_{ab} is minimal over all such symmetric pairs, so $e_{pq}\geq e_{ab}$. If the strict inequality $F(v)>e_{ab}$ were to hold, then let $\varepsilon=\frac{1}{2}(F(v)-e_{ab})$ and choose M_1 with

$$|v(-M_1) - \mathbf{p}|^2 < \frac{\varepsilon}{8C_0},$$

and constant C_0 as in Lemma 2.1. For any constant $M > 0$, there exists y_M such that

$$|v(x_1) - U(x)| \leq \frac{\varepsilon}{4C_0}, \quad \forall x \in [-M_1, M_1] \times [y_M, y_M + M] \subset T_{y_M+M}.$$

Then, from Lemma 2.1 we obtain the following lower bound:

$$\begin{aligned} C_1 &\geq \int_{T_{y_M+M}\setminus T_{y_M}} \left(\left| \frac{\partial U}{\partial x_1} \right|^2 + W(U) \right) dx - e_{ab} M \\ &\geq M(F(v) - \varepsilon - e_{ab}) > M\varepsilon. \end{aligned}$$

Since $M > 0$ is arbitrary, this contradicts the upper bound obtained in Theorem 6.2. Hence, v is a minimizer of F among curves connecting $\mathbf{p}, \mathbf{q}$ with $F(v) = e_{pq} = e_{ab}$.

To show that v must in fact connect $\mathbf{a}, \mathbf{b}$, we use an argument similar to the lower bound of Proposition 3.1 or Lemma 4.5 of [3]. Let $\delta > 0$ be fixed with $\delta < \frac{1}{2}\|v - z\|_\infty$ for all $z \in \mathscr{Z}_{ab}$. By the convergence $U_L \to U$, for every $M > 0$ there exists L such that

$$|U_L(x_1, y_M) - v| < \frac{\delta}{8}$$

for all $|x_1| \leq M$ and some $y_M \in (M, L)$. On the other hand, $U_L(x_1, L) = z(x_1)$ at $x_2 = L$, so $U_L(\cdot, x_2)$ must eventually exit a neighborhood of v, then enter a neighborhood of some $z \in \mathscr{Z}_{ab}$. As in the proof of Proposition 3.1, there exist y_M^1 and y_M^2 with $M < y_M^1 < y_M^2 < L$, $z \in \mathscr{Z}_{ab}$, and $\mu > 0$ (independent of L, M) such that:

$$\max_{|x_1|\leq M} |U_L(x_1, y_M^1) - v(x)| < \frac{\delta}{4}, \quad \max_{|x_1|\leq M} |U_L(x_1, y_M^2) - z(x)| < \frac{\delta}{4}, \tag{6.9}$$

$$\int_{-x_2}^{x_2} \frac{1}{2} \left(\left| \frac{\partial U_L}{\partial x_1} \right|^2 + W(U_L) \right) dx_1 > e_{ab} + \mu, \quad \forall x_2 \in (y_M^1, y_M^2). \tag{6.10}$$

(The inequality (6.10) rests on a finite-interval version of Lemma 2.4. See Proposition 3.9 of [3].)

We may now estimate the energy from below:

$$\begin{aligned} C_1 &\geq \int_{T_{y_M^2}\setminus T_{y_M^1}} \left(\left| \frac{\partial U_L}{\partial x_1} \right|^2 + W(U_L) \right) dx_1 - e_{ab}(y_M^2 - y_M^1) \\ &\geq \mu(y_M^2 - y_M^1) + \int_{T_{y_M^2}\setminus T_{y_M^1}} \frac{1}{2} \left| \frac{\partial U_L}{\partial x_2} \right|^2 dx_2\, dx_1 \\ &\geq \mu(y_M^2 - y_M^1) + \int_{-M}^{M} \int_{y_M^1}^{y_M^2} \frac{1}{2} \left| \frac{\partial U_L}{\partial x_2} \right|^2 dx_2\, dx_1 \\ &\geq \mu(y_M^2 - y_M^1) + \frac{\delta}{4} \frac{2M}{y_M^2 - y_M^1} \\ &\geq \delta\sqrt{2\mu M}. \end{aligned}$$

[Note that the integral of $|\frac{\partial U_L}{\partial x_2}|^2$ is estimated as in (3.2).] Since M is arbitrary, we obtain a contradiction with Theorem 6.2. We conclude that $v \in \mathscr{S}_{ab}$, and since $F(v) = e_{ab}$ moreover $v \in \mathscr{Z}_{ab}$.

It remains to show that the limit function v is unique (and hence independent of the choice of a subsequence $y_n \to \infty$.) By hypothesis, the set of minimizers $\mathscr{Z}_{ab}$ has only finitely many elements $z_1, \ldots, z_N$, and hence we may conclude that

$$\delta = \inf_{i \neq j} \|z_i - z_j\|_\infty > 0.$$

Choose $M > 0$ large enough such that $\inf_{i \neq j} \|z_i - z_j\|_{L^\infty(-M,M)} > \delta/2$. If $U(x_1, x_2)$ were to have distinct limits $z, \tilde{z} \in \mathscr{Z}_{ab}$ along subsequences $x_{2,n}, \tilde{x}_{2,n} \to +\infty$, then we could find another subsequence $y_n \to +\infty$ (y_n lying between $x_{2,n}, \tilde{x}_{2,n}$) with $\|U(\cdot, y_n) - z_i\|_{L^\infty(-M,M)} \geq \delta/4$ for each $i = 1, \ldots, N$. But, repeating the above arguments with the sequence y_n would then yield a contradiction, as a further subsequence $U(x_1, y_{n_k})$ would then converge to an element of $\mathscr{Z}_{ab}$ in $L^\infty(-M, M)$.

In conclusion, $\lim_{x_2 \to +\infty} U(x_1, x_2) = z(x_1)$. Finally, by repeating the analysis of Theorem 4.7 of [3] we may prove that the above limit is *uniform* in x_1. By symmetry of the solution U we may conclude that the corresponding limit conditions (6.2)–(6.4) are also satisfied, and U is the desired saddle solution.

Remark 6.3 Note that if $\mathscr{Z}_{ab}$ consists of a single trajectory $z(t)$ the above construction may be made without imposing symmetry with respect to the diagonal of the square, γ''. Generally, if γ''-invariance is not imposed on the admissible class $\tilde{\mathscr{M}}$ it is possible that U tends to distinct minimizing curves in different directions: $U(x_1, x_2) \to z_1(\pm x_1)$ as $x_2 \to \pm\infty$ but $U(x_1, x_2) \to z_2(\pm x_2)$ as $x_1 \to \pm\infty$ with $z_1, z_2 \in \mathscr{Z}_{ab}$ different minimizing curves for F in $\mathscr{S}_{ab}$.

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