On the Slowness of Phase Boundary Motion in One Space Dimension

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Abstract

We study the limiting behavior of the solution of

$$u_t - \epsilon^2 u_{xx} + u^3 - u = 0, \qquad a < x < b,$$

with a Neumann boundary condition or an appropriate Dirichlet condition. The analysis is based on "energy methods". We assume that the initial data has a "transition layer structure", i.e., $u^\epsilon \approx \pm 1$ except near finitely many transition points. We show that, in the limit as $\epsilon \to 0$, the solution maintains its transition layer structure, and the transition points move slower than any power of ϵ.

1. Introduction

We study the initial-value problem

$$u_t - \epsilon^2 u_{xx} + u^3 - u = 0, \qquad a < x < b, \tag{1.1}$$

with either a homogeneous Neumann boundary condition,

$$u_x(a, t) = u_x(b, t) = 0, \qquad t > 0, \tag{1.2}$$

or else a Dirichlet condition of the form

$$u(a, t) = \alpha, \quad u(b, t) = \beta, \quad t > 0, \quad \alpha, \beta = \pm 1. \tag{1.3}$$

Our interest is in the limiting behavior of $u = u^\epsilon$ as $\epsilon \to 0$. The initial data are assumed to have a "transition layer structure". The precise hypotheses are given in Section 3; the main point is that $u^\epsilon(x, 0) \approx \pm 1$ except near finitely many transition points (see Figure 1). It turns out that u^ϵ *maintains its transition layer structure, and the transition points move very slowly as* $\epsilon \to 0$. Our main results, Theorems 4.1 and 4.3, assert roughly speaking that the transitions move slower than any power of ϵ.

Our work is closely related to that of Neu [22], Carr and Pego [5],[6], and Fusco and Hale [11],[12]. Neu uses the method of matched asymptotic expansions

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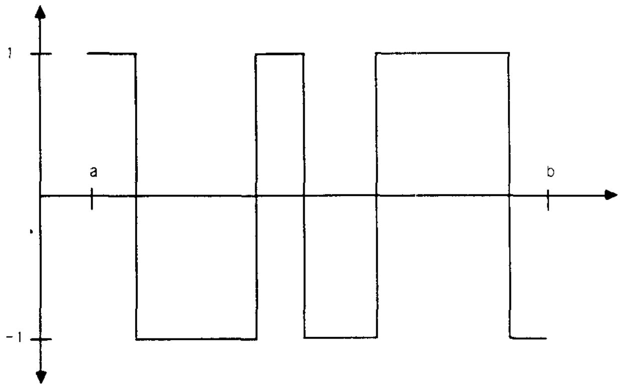

Figure 1.

to predict the form of $u^\epsilon(x, t)$, and in particular to predict the velocities of the transition points. The papers of Carr and Pego and Fusco and Hale give rigorous justifications of Neu's formal analysis when u satisfies the Neumann boundary condition (1.2). According to these results the transition points move with velocities of order $e^{-Cl/\epsilon}$, where C is a constant and l is the minimum distance between the transitions in the initial data. Our conclusion is weaker, since $e^{-Cl/\epsilon} \ll \epsilon^k$ for any k as $\epsilon \to 0$. However our analysis has several advantages over those of [6],[12]: (1) it is far more elementary, being based entirely on "energy-type" estimates; (2) it handles both boundary conditions (1.2) and (1.3) with equal ease; and (3) it places less stringent requirements on the form of the initial data. Moreover, our method provides a rather clear and intuitive explanation as to why the evolution should be so slow.

The multidimensional analogue of (1.1), $u_t - \epsilon^2\Delta u + u^3 - u = 0$, has also been the object of recent attention. The situation is rather different in R^n, $n \geqq 2$, since then u^ϵ has "transition surfaces" rather than "transition points". As $\epsilon \to 0$, the transition surface is believed to move with velocity $\epsilon^2\kappa$, where κ is the sum of the principal curvatures. This was first (to our knowledge) observed by Allen and Cahn in [1], based on a formal calculation. A much more detailed analysis, still formal in character, has been given by Rubinstein, Sternberg, and Keller in [15]. We have proved the validity of this conjecture for certain *radial* solutions in [4], using a method very close to that of the present paper. A rigorous result without radial symmetry has recently been announced by De Mottoni and Schatzmann; see [7] and [25]. The weaker assertion that the transition layer moves with velocity $o(\epsilon)$ can be deduced from the work of Freidlin and Gärtner; see [10],[13].

There is a specific, physical motivation for studying (1.1): the multidimensional version was proposed in [1] as a model for the motion of antiphase boundaries in crystalline solids. Similar equations also arise in other contexts: this is a "type A" Ginzburg-Landau equation, in the terminology of [13].

There is also a broader and very fundamental reason to study (1.1), namely its status as an example of *dynamical metastability*. It is well known that as $t \to \infty$ any solution of (1.1) is asymptotically stationary; for generic initial data u tends to a local minimum of the "energy"

$$\int_a^b \left[\frac{\epsilon^2}{2} u_x^2 + \frac{1}{4}(u^2-1)^2\right] dx; \tag{1.4}$$

see for example [18]. However our results show that this convergence is exceedingly slow: on a time scale of order ϵ^{-k} nothing happens! The transition layer function in Figure 1 is not a local minimum of (1.4), nor even a stationary point; however when ϵ is small it may well *appear* stationary to a casual observer. It is common, especially in elasticity, to focus attention on local minimizers of energy as being the only observable stationary states. We learn from (1.1) that this can be dangerous: if small effects (such as surface energy) are present, then there may well be non-stationary states which persist for a very long time.

The analyses of Carr and Pego and Fusco and Hale are based on the use of an ansatz for the form of u^ϵ, and on estimates for the linearization of (1.1) about this ansatz. Our method is totally different, much closer in spirit to recent studies of the stationary problem which make use of the notion of Γ-convergence ([9],[16],[17],[19]–[21],[23],[24]). It is convenient to normalize the energy so as to keep it positive and finite as $\epsilon \to 0$; we therefore set

$$E_\epsilon[v] = \int_a^b \left[\frac{\epsilon}{2} v_x^2 + \frac{1}{4\epsilon}(v^2-1)^2\right] dx \tag{1.5}$$

for any $v : (a, b) \to R$. When specialized to one space dimension, the results of [19],[21], or [23] assert that the minimum energy of a transition between $+1$ and -1 is asymptotically

$$c_0 = \frac{1}{\sqrt{2}} \int_{-1}^{+1} (1-s^2)\, ds = \frac{2\sqrt{2}}{3}. \tag{1.6}$$

In other words, if $\{v^\epsilon\}$ converges in $L^1((a, b))$ to a limit v^0, and if v^0 makes N transitions between $+1$ and -1, then

$$\liminf_{\epsilon \to 0} E_\epsilon[v^\epsilon] \geqq Nc_0, \tag{1.7}$$

with equality if the sequence $\{v^\epsilon\}$ is chosen properly. An essential step in our argument is an improvement of this result, what amounts to an error estimate for (1.7): we shall show that for any $k > 0$

$$E_\epsilon[v^\epsilon] \geqq Nc_0 - C\epsilon^k \tag{1.8}$$

when ϵ is sufficiently small. (See Proposition 2.1.)

We now explain briefly why solutions of (1.1) with appropriately chosen initial data must evolve slowly. It is easy to verify that

$$E_\epsilon[u^\epsilon](0) - E_\epsilon[u^\epsilon](T) = \epsilon^{-1} \int_0^T \int_a^b (u_t^\epsilon)^2 \, dx \, dt \tag{1.9}$$

(see the proof of Proposition 3.1). Suppose that the initial data makes N transitions between $+1$ and -1, and satisfies

$$E_\epsilon[u^\epsilon](0) \leqq c_0 N + \epsilon^k. \tag{1.10}$$

If we choose $T = T_\epsilon$ so that $u^\epsilon(\cdot, T_\epsilon)$ still makes N transitions, then (1.8) suggests that

$$E_\epsilon[u^\epsilon](T_\epsilon) \geqq c_0 N - C\epsilon^k, \tag{1.11}$$

and substitution into (1.9) gives

$$\int_0^{T_\epsilon} \int_a^b (u_t^\epsilon)^2 \, dx \, dt \leqq C\epsilon^{k+1}. \tag{1.12}$$

If k is large and T_ϵ is not small then this forces u_t^ϵ to be small, at least in the mean, as $\epsilon \to 0$. There are, of course, some technicalities: for example we will have to prove that $T_\epsilon \gg 0$. Nevertheless (1.9)–(1.12) show quite clearly why the evolution is so slow. The point is that the velocity u_t^ϵ is linked to the dissipation of energy by (1.9), while most of the energy is due to the mere *existence* of the transitions, regardless of their location, by (1.8). There is very little "excess" energy to be dissipated by the motion of the transitions, and so their evolution is very slow.

A word is in order about the initial data. The analyses of Carr and Pego and Fusco and Hale assume that $u^\epsilon(x, 0)$ is close to a suitable ansatz; such u^ϵ will automatically satisfy (1.10) for every $k > 0$, so our hypotheses on the initial data are no more restrictive than theirs. Significantly, however, we still obtain a result, even if (1.10) is satisfied only up to some finite value of k: then the transition points move with velocity $O(\epsilon^{k+1})$. The ability to handle simultaneously this larger class of initial data is one of the advantages of our method. It should be emphasized, however, that like [6],[12] we are obliged to consider initial data which depend on ϵ. Of the various methods yielding rigorous results about (1.1) as $\epsilon \to 0$, only that of Freidlin and Gärtner is free from this deficiency; see [10],[13]. (But see [8] for results on the Cauchy problem for (1.1) with ϵ-independent initial data.)

Our attention is focused on the specific equation (1.1) only for the sake of simplicity. In fact, our results extend quite easily to the more general equation

$$u_t - \epsilon^2 u_{xx} + F'(u) = 0$$

where F is a bistable potential with both wells of equal depth. The case when u is vector-valued or F has more than two wells (always of equal depth) can be handled similarly, albeit with more effort, by using the methods of [24], [9], or [3].

This work was strongly influenced by discussions with S. Luckhaus and R. Pego during the fall of 1987. It is a pleasure to acknowledge their significant role in the development of these ideas.

2. A Lower Bound on the Energy

This section presents our lower bound on the energy due to the presence of N transitions. The result is purely variational in character: the evolution equation (1.1) plays no role.

We fix for the remainder of this section an integer $N \geqq 1$, and a piecewise constant function $v:(a,b) \to \{+1,-1\}$ with exactly N discontinuities. The "energy" E_ϵ is defined by (1.5), and the constant c_0 is given by (1.6).

PROPOSITION 2.1. *Let l be a positive integer. There exist constants $\delta_l > 0$ and $c_l > 0$ with the following property: if w is an H^1 function on (a, b) satisfying*

$$\int_a^b |w - v| \, dx \leqq \delta_l \tag{2.1}$$

and

$$E_\epsilon[w] \leqq Nc_0 + \epsilon^l \tag{2.2}$$

with $\epsilon \leqq 1$, then in fact

$$E_\epsilon[w] \geqq Nc_0 - c_l\epsilon^l. \tag{2.3}$$

Proof: We consider first the case $N = 1$. The idea of the argument is this: using (2.1) and (2.2) we can select points where w is close to $+1$ and -1; applying the method of [21], [19], or [23] this leads to (2.3) with $l = 1$. That in turn allows us to select new points where w is closer to ± 1, leading inductively to (2.3) for any l.

Let γ be the point of discontinuity of v. Replacing v and w by $-v$ and $-w$ if necessary, we may assume that $v \equiv -1$ on (a, γ). We choose δ_l so that

$$(\gamma - 2l\delta_l, \gamma + 2l\delta_l) \subset (a, b). \tag{2.4}$$

We assert the existence of points $x_1 \in (\gamma - 2\delta_l, \gamma)$ and $y_1 \in (\gamma, \gamma + 2\delta_l)$ such that

$$w(x_1) \leqq -1 + C\epsilon^{1/2}, \qquad w(y_1) \geqq 1 - C\epsilon^{1/2}. \tag{2.5}$$

(Here and throughout, C represents a constant that is independent of ϵ, whose value may change from line to line.) Indeed, from (2.1)

$$\int_a^\gamma |w+1|\,dx \leqq \delta_l. \tag{2.6}$$

Setting

$$S^+ = \{x : w(x) \geqq 0\}, \qquad S^- = \{x : w(x) < 0\}, \tag{2.7}$$

it follows from (2.6) that $\operatorname{meas}(S^+ \cap (a, \gamma)) \leqq \delta_l$, and hence that

$$\operatorname{meas}(S^- \cap (\gamma - 2\delta_l, \gamma)) \geqq \delta_l. \tag{2.8}$$

Now, (2.2) yields

$$\int_{S^- \cap (\gamma - 2\delta_l, \gamma)} \frac{1}{4}\epsilon^{-1}(w^2-1)^2\,dx \leqq c_0 + 1, \tag{2.9}$$

whence there exists $x_1 \in S^- \cap (\gamma - 2\delta_l, \gamma)$ such that

$$(w^2(x_1) - 1)^2 \leqq C\epsilon, \qquad C = \frac{4(c_0+1)}{\delta_l}. \tag{2.10}$$

Since $w(x_1) < 0$, (2.10) entails

$$w(x_1) \leqq -1 + C\epsilon^{1/2}.$$

The existence of $y_1 \in S^+ \cap (\gamma, \gamma + 2\delta_l)$ such that $w(y_1) \geqq 1 - C\epsilon^{1/2}$ is proved similarly.

Next we prove that

$$\int_{x_1}^{y_1} \left[\frac{\epsilon}{2}|w_x|^2 + \frac{1}{4\epsilon}(w^2-1)^2\right] dx \geqq c_0 - c_1\epsilon \tag{2.11}$$

as a consequence of (2.5). Since $a^2 + b^2 \geqq 2\,|a|\,|b|$, the left side of (2.11) is bounded below by

$$\frac{1}{\sqrt{2}}\int_{x_1}^{y_1} |w_x|\,|w^2-1|\,dx = \int_{x_1}^{y_1} \left|\frac{d}{dx}g(w)\right| dx, \tag{2.12}$$

where $g'(s) = (1/\sqrt{2})|s^2 - 1|$. Note that the constant c_0, defined by (1.6), is

$$c_0 = g(1) - g(-1).$$

Using the monotonicity of g we conclude that

$$(2.12) \geqq g(w(y_1)) - g(w(x_1))$$

$$\geqq c_0 - \int_{1-C\epsilon^{1/2}}^{1} \frac{1}{\sqrt{2}}(1-s^2)\,ds - \int_{-1}^{-1+C\epsilon^{1/2}} \frac{1}{\sqrt{2}}(1-s^2)\,ds$$

$$(2.13) \qquad \geqq c_0 - c_1\epsilon,$$

establishing (2.11). Note that this implies (2.3) when $l = 1$.

Now we argue inductively to prove the following assertions for $1 \leqq k \leqq l$:

There exist $x_k \in (\gamma - 2k\delta_l, \gamma)$ and $y_k \in (\gamma, \gamma + 2k\delta_l)$ such that

(2.14) $\quad w(x_k) \leqq -1 + C\epsilon^{k/2}$ and $w(y_k) \geqq 1 - C\epsilon^{k/2}$, with $C = C(k)$ independent of ϵ;

$$(2.15) \qquad \int_{x_k}^{y_k} \left[\frac{\epsilon}{2}|w_x|^2 + \frac{1}{4\epsilon}(w^2-1)^2\right] dx \geqq c_0 - c_k\epsilon^k.$$

We have already completed the initial step, $k = 1$. Let us show that for $k < l$, $(2.15)_k \Rightarrow (2.14)_{k+1}$. Indeed, $(2.15)_k$ and (2.2) yield

$$(2.16) \qquad \int_a^{x_k} \frac{1}{4\epsilon}(w^2-1)^2\,dx \leqq C\epsilon^k,$$

from which it follows that

$$(2.17) \qquad \int_{S^- \cap (x_k - 2\delta_l, x_k)} (w^2-1)^2\,dx \leqq C\epsilon^{k+1},$$

with S^- as in (2.7). By (2.1),

$$(2.18) \qquad \operatorname{meas}(S^- \cap (x_k - 2\delta_l, x_k)) \geqq \delta_l.$$

Arguing as for (2.5) but using (2.17) and (2.18) in place of (2.9) and (2.8), we conclude the existence of $x_{k+1} \in (x_k - 2\delta_l, x_k)$ with the desired property. The existence of $y_{k+1} \in (y_k, y_k + 2\delta_l)$ is proved similarly.

To complete the induction we must show that $(2.14)_{k+1} \Rightarrow (2.15)_{k+1}$ if $k < l$. The argument is entirely parallel to (2.12)–(2.13):

$$\int_{x_k}^{y_k} \left[\frac{\epsilon}{2}|w_x|^2 + \frac{1}{4\epsilon}(w^2-1)^2\right] dx \geqq \int_{x_k}^{y_k} \left|\frac{d}{dx} g(w)\right| dx$$

$$\geqq g(y_k) - g(x_k)$$

$$\geqq c_0 - c_{k+1}\epsilon^{k+1},$$

making use of $(2.14)_{k+1}$ in the last step. Since (2.15) with $k = l$ implies (2.3), we have completed the proof in case $N = 1$.

The preceding argument is fundamentally local in character, so it is readily adapted to handle the case $N \geqq 2$. Let v have discontinuities at points $\gamma_1 < \gamma_2 < \cdots < \gamma_N$; for ease of notation we set $a = \gamma_0$, $\gamma_{N+1} = b$. The constant δ_l should now be chosen so that

$$\gamma_i + 2l\delta_l < \gamma_{i+1} - 2l\delta_l \qquad 0 \leqq i \leqq N. \tag{2.19}$$

We may assume without loss of generality that $v \equiv -1$ on (a, γ_1). Arguing as for (2.5) we obtain the existence of points $x_1^i \in (\gamma_i - 2\delta_l, \gamma_i)$ and $y_1^i \in (\gamma_i, \gamma_i + 2\delta_l)$ such that

$$\begin{aligned} w(x_1^i) &\approx (-1)^i, & w(y_1^i) &\approx (-1)^{i+1}, \\ (w(x_1^i)^2 - 1)^2 &\leqq C\epsilon, & (w(y_1^i)^2 - 1)^2 &\leqq C\epsilon. \end{aligned}$$

The energy on each interval (x_1^i, y_1^i) can be estimated as in (2.11); adding these estimates gives

$$E_\epsilon[w] \geqq Nc_0 - c_1\epsilon,$$

which is (2.3) when $l = 1$. An inductive argument entirely parallel to that given earlier leads to (2.3) for the general case $l \geqq 2$.

Remark 2.2. Examination of the proof shows that the constants δ_l and c_l depend on v only through the number of its transitions and their distances from one another and from the endpoints of the interval.

3. A Bound on the Time Derivative

We now turn to the initial value problem (1.1) with boundary condition (1.2) or (1.3). The initial data should have a "transition layer structure"; we give this a precise meaning as follows. On the one hand, we suppose that

$$v(x) = \lim_{\epsilon \to 0} u^\epsilon(x, 0) \tag{3.1}$$

exists as a limit in the L^1 norm, and that v is a piecewise constant function taking only the values ± 1, with exactly N discontinuities. In addition, we suppose that for all sufficiently small ϵ

$$E_\epsilon[u^\epsilon](0) \leqq Nc_0 + C\epsilon^k, \tag{3.2}$$

with c_0 as in (1.6), and k a positive integer. The first condition fixes the number of transitions in the initial data and their relative positions as $\epsilon \to 0$. The second one demands that $u^\epsilon(x, 0)$ make these transitions "efficiently," i.e., with excess energy at most $C\epsilon^k$ over the minimum possible (c.f. Proposition 2.1). Such initial data are easily constructed for any choice of v; indeed, by arguing as in [23] one can choose u^ϵ to satisfy (3.2) for *every* $k > 0$ provided that $\epsilon < \epsilon_0(k)$. To simplify notation we often write u instead of u^ϵ.

PROPOSITION 3.1. *Assume that the initial data $u^\epsilon(x, 0)$ satisfy (3.1) and (3.2) for some choice of $k > 0$. Then there exist constants F and G such that*

$$\int_0^{F\epsilon^{-(k+1)}} \int_a^b (u_t^\epsilon)^2 \, dx \, dt \leqq G\epsilon^{k+1} \tag{3.3}$$

for all sufficiently small ϵ. The values of F and G depend on v and k, but not on ϵ.

Proof: By (3.1) the relation

$$\int_a^b |u^\epsilon(x, 0) - v(x)| \, dx \leqq \frac{1}{2} \delta_k \tag{3.4}$$

holds for all sufficiently small ϵ, with δ_k as in Proposition 2.1. We henceforth consider only values of ϵ for which (3.2) and (3.4) hold.

Multiplying (1.1) by u_t, integrating, then integrating by parts using either (1.2) or (1.3), we have

$$\epsilon^{-1} \int_0^T \int_a^b u_t^2 \, dx \, dt = E_\epsilon[u](0) - E_\epsilon[u](T) \tag{3.5}$$

for any $T > 0$. It follows that $E_\epsilon[u]$ decreases in time, and that u_t^2 is integrable.

We assert that if $T = T(\epsilon)$ satisfies

$$\int_0^{T(\epsilon)} \int_a^b |u_t^\epsilon| \, dx \, dt \leqq \frac{1}{2} \delta_k \tag{3.6}$$

then

$$\int_0^{T(\epsilon)} \int_a^b (u_t^\epsilon)^2 \, dx \, dt \leqq C\epsilon^{k+1}. \tag{3.7}$$

Indeed, if (3.6) holds then $w(x) = u^\epsilon(x, T(\epsilon))$ satisfies

$$\begin{aligned} \int_a^b |v - w| \, dx &\leqq \int_a^b |v(x) - u^\epsilon(x, 0)| \, dx + \int_a^b |u^\epsilon(x, 0) - w(x)| \, dx \\ &\leqq \frac{1}{2} \delta_k + \frac{1}{2} \delta_k \end{aligned}$$

by (3.4) and (3.6), so by Proposition 2.1

$$E_\epsilon[u](T(\epsilon)) = E_\epsilon[w] \geqq Nc_o - c_k\epsilon^k. \tag{3.8}$$

Substitution of (3.8) and (3.2) in (3.5) yields

$$\epsilon^{-1}\int_0^{T(\epsilon)}\int_a^b u_t^2\,dx\,dt \leqq (c_k+1)\epsilon^k,$$

which gives (3.7).

Thus to prove (3.3) we must simply show that (3.6) holds with $T(\epsilon) \geqq C\epsilon^{-(k+1)}$. If

$$\int_0^\infty\int_a^b |u_t^\epsilon|\,dx\,dt \leqq \frac{1}{2}\delta_k$$

then there is nothing to prove; otherwise choose $T_1(\epsilon)$ such that

$$\int_0^{T_1(\epsilon)}\int_a^b |u_t^\epsilon|\,dx\,dt = \frac{1}{2}\delta_k.$$

By (3.7) we have

$$\begin{aligned}\frac{1}{2}\delta_k &\leqq C(T_1(\epsilon))^{1/2}\left(\int_0^{T_1(\epsilon)}\int_a^b |u_t^\epsilon|^2\,dx\,dt\right)^{1/2}\\ &\leqq C(T_1(\epsilon))^{1/2}\epsilon^{(k+1)/2},\end{aligned}$$

so

$$T_1(\epsilon) \geqq C\epsilon^{-(k+1)},$$

as desired.

4. The Transitions Move Slowly

It remains to deduce from Proposition 3.1 an estimate for the rate at which the profile of u changes. The easiest conclusion is that "nothing happens on a time scale of order ϵ^{-k}":

THEOREM 4.1. *Assume that the initial data $u^\epsilon(x,0)$ satisfy (3.1) and (3.2) for some $k>0$. Then for any $m>0$*

$$\lim_{\epsilon\to 0}\sup_{0\leqq t\leqq\epsilon^{-k}m}\int_a^b |u^\epsilon(x,t)-v(x)|\,dx = 0. \tag{4.1}$$

Proof: Let $\tilde{u}^\epsilon(x, \tau) = u^\epsilon(x, \epsilon^{-k}\tau)$. Restated in terms of $\tilde{u}^\epsilon$, Proposition 3.1 asserts that

$$\int_0^{F\epsilon^{-1}} \int_a^b (\tilde{u}^\epsilon_\tau)^2 \, dx \, d\tau \leqq G\epsilon.$$

Thus by Hölder's inequality

$$\int_0^m \int_a^b |\tilde{u}^\epsilon_\tau| \, dx \, d\tau \leqq m^{1/2}(b-a)^{1/2}(G\epsilon)^{1/2}, \tag{4.2}$$

provided that ϵ is small enough so that $F\epsilon^{-1} \geqq m$. Now,

$$\sup_{0 \leqq \tau \leqq m} \int_a^b |\tilde{u}^\epsilon(x, \tau) - \tilde{u}^\epsilon(x, 0)| \, dx \leqq \int_0^m \int_a^b |\tilde{u}^\epsilon_\tau| \, dx \, d\tau, \tag{4.3}$$

and (3.1) gives

$$\lim_{\epsilon \to 0} \int_a^b |\tilde{u}^\epsilon(x, 0) - v(x)| \, dx = 0. \tag{4.4}$$

A combination of (4.2), (4.3), and (4.4) gives

$$\lim_{\epsilon \to 0} \sup_{0 \leqq \tau \leqq m} \int_a^b |\tilde{u}^\epsilon(x, \tau) - v(x)| \, dx = 0,$$

which is equivalent to (4.1).

It is natural to ask for a statement which discusses the motion of the "transition points" more directly. To do so we make a modest extra hypothesis on the initial data:

(4.5) $u^\epsilon(x, 0)$ crosses 0 transversely, at exactly N distinct points.

By [2] the same is then true of $u^\epsilon(x, t)$ for $t > 0$, until the first time when two of its zeros coalesce or one of them reaches an endpoint of the interval. Our goal is Theorem 4.3, which estimates the time it takes for one of the zeros to move a small but appreciable amount. However a preliminary lemma is required first concerning the structure of $u^\epsilon(\cdot, t)$. Like the results in Section 2, this lemma really has nothing to do with the dynamics, and so we cast it as a result about a general H^1 function $w: (a, b) \to R$.

LEMMA 4.2. *Suppose that the graph of w crosses 0 transversely, at exactly N points $z_1 < \cdots < z_n$. Assume moreover that $E_\epsilon[w] \leqq c_0 N + C\epsilon$. Then for δ sufficiently small and $\epsilon \leqq \epsilon_0(\delta)$ there exist intervals (x_i, y_i) containing z_i such that*

$$|x_i - z_i| \leqq \delta, \qquad |y_i - z_i| \leqq \delta \tag{4.6}$$

$$(w^2(x) - 1)^2 \leqq \delta \quad \text{for} \quad x \notin \bigcup_{i=1}^{N} (x_i, y_i). \tag{4.7}$$

Proof: The asserted profile of w is shown in Figure 2. Arguing as for (2.5) and (2.11) we easily obtain the existence of points x_i, y_i satisfying (4.6) and

$$(w^2(x_i) - 1)^2 \leqq C\epsilon/\delta, \qquad (w^2(y_i) - 1)^2 \leqq C\epsilon/\delta, \tag{4.8}$$

$$\int_{x_i}^{y_i} \left[\frac{\epsilon}{2} w_x^2 + \frac{1}{4\epsilon}(w^2 - 1)^2\right] dx \geqq c_o - C\epsilon/\delta. \tag{4.9}$$

Let $V = (a, b)\backslash \cup_{i=1}^{N}(x_i, y_i)$. Then by (4.9)

$$\begin{aligned} \int_V \left[\frac{\epsilon}{2} w_x^2 + \frac{1}{4\epsilon}(w^2 - 1)^2\right] dx &\leqq E_\epsilon[w] - Nc_o + NC\epsilon/\delta \\ &\leqq C\epsilon/\delta \end{aligned}$$

Arguing as in (2.12) we conclude that

$$\int_V \left|\frac{d}{dx} g(w)\right| dx \leqq C\epsilon/\delta.$$

Thus on each connected component of V the oscillation of $g(w)$ is controlled. The endpoints are controlled as well, by (4.8). When ϵ is sufficiently small these estimates yield (4.7).

THEOREM 4.3. *Assume that the initial data $u^\epsilon(x, 0)$ satisfy (3.1), (3.2), and (4.5). Let*

$$z_1^\epsilon(t) < z_2^\epsilon(t) < \cdots < z_N^\epsilon(t)$$

be the zeros of $u^\epsilon(t)$. Given $\delta_1 > 0$ let

$$T_\epsilon(\delta_1) = \inf\{t : |z_i^\epsilon(t) - z_i^\epsilon(0)| > \delta_1 \quad \textit{for some} \quad i\}.$$

If δ_1 is sufficiently small then

$$T_\epsilon(\delta_1) \geqq C\delta_1^2 \epsilon^{-(k+1)} \tag{4.10}$$

for $\epsilon < \epsilon_0(\delta_1)$, with C independent of δ_1 as well as ϵ.

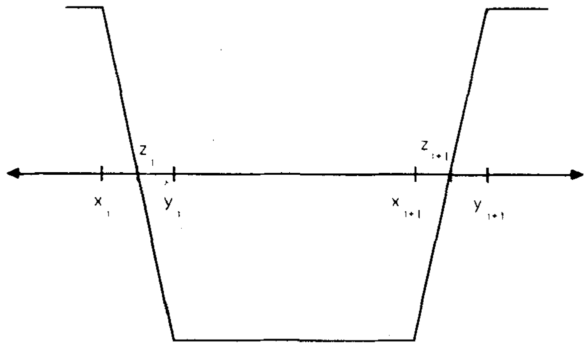

Figure 2.

Proof: If $T_\epsilon(\delta_1) = \infty$ then there is nothing to prove, so we shall assume that $T_\epsilon(\delta_1) < \infty$. Since $z_i^\epsilon(t)$ is continuous in t,

$$|z_i^\epsilon(T_\epsilon(\delta_1)) - z_i^\epsilon(0)| = \delta_1 \quad \text{for some} \quad i.$$

We claim that this implies

$$\int_a^b |u^\epsilon(x, T_\epsilon(\delta_1)) - u^\epsilon(x, 0)|\, dx \geqq C\delta_1, \tag{4.11}$$

provided that ϵ is small enough. Indeed, by Lemma 4.2 (with $\delta \ll \delta_1$ and $w(x) = u^\epsilon(x, T_\epsilon(\delta_1))$), $u^\epsilon(\cdot, T_\epsilon(\delta_1))$ is essentially a step function with discontinuities at $z_i^\epsilon(T_\epsilon(\delta_1))$. Similarly, $u^\epsilon(x, 0)$ is essentially a step function with discontinuities at $z_i^\epsilon(0)$. This leads easily to (4.11).

Now, (4.11) implies that

$$C\delta_1 \leqq \int_0^{T_\epsilon(\delta_1)} \int_a^b |u_t^\epsilon|\, dx\, dt$$

$$\leqq \left(\int_0^{T_\epsilon(\delta_1)} \int_a^b (u_t^\epsilon)^2\, dx\, dt \right)^{1/2} (b-a)^{1/2} T_\epsilon(\delta_1)^{1/2}.$$

Let F be the constant introduced in Proposition 3.1. If $T_\epsilon(\delta_1) \geqq F\epsilon^{-(k+1)}$, then (4.10) follows trivially; otherwise, if $T_\epsilon(\delta_1) \leqq F\epsilon^{-(k+1)}$ we bound the integral using (3.3) to get

$$\delta_1 \leqq C\epsilon^{(k+1)/2} T_\epsilon(\delta_1)^{1/2},$$

which yields (4.10).

Thus according to Theorem 4.3 one must wait a time of order $\epsilon^{-(k+1)}$ to see an appreciable change in the position of the zeros of u^ϵ.

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