Lecture 1: Introduction to functions

A *function* is a rule that assigns each elements in a set A to *exactly one* object in a set B. The set A is called the *domain* of a function (the set of input) and the set B is the called the *range* of the function (the set of output).

In this course, we will only deal with functions whose domain is a subset of real numbers. Usually we will denote it by $y = f(x)$, where x is the input and is called the *independent variable* and y is the output and is called the *dependent variable.* By convention, we will usually say the domain is the largest set for which the function can be defined (with the output are finite numbers).

Examples. (i) $f(x) = x^2 + 1$, then the domain of f is every real number. $f(1) = 1^1 + 1 = 2$ and $f(x+1) = (x+1)^2 + 1 = x^2 + 2x + 2$.

(ii) If $g(x) = \frac{1}{x-2}$, then $g(x)$ is undefined at $x = 2$ so the domain is every real number except 2. $g(3) = \frac{1}{3-2} = 1$.

(iii) If $h(x) = (x+3)^{1/2} = \sqrt{x+3}$, then the domain of h is $x+3 \geq 0$ (i.e. $x \geq -3$).

We can also define the *composition* of functions $f \circ g(x) = f(g(x))$. For example, if f and g are in Examples (i) and (ii) above, then $f \circ g(x) = f(g(x)) = \left(\frac{1}{x-2}\right)^2 + 1 = \frac{x^2-4x+5}{(x-2)^2}$ and $g \circ f(x) = \frac{1}{x^2+1-2} = \frac{1}{x^2-1}$.

We can sketch the *graph* of function $y = f(x)$ in the coordinates system. The graph on the coordinates system is a function if and only if every vertical line cuts the graph at most one point. This is called the *vertical line test.*

The *y-intercept* of a function $f(x)$ is the point $f(0)$. The *x-intercept* of a function $f(x)$ is value of x in the domain such that $f(x) = 0$.

Examples: We discuss the examples of functions we will encounter in this course.

1. **Linear function (Straight line)**: $y = f(x) = mx + c$, where m is the slope and c is the y-intercept. If $m = 0$, it is the constant function.

2. **Parabola:** $f(x) = ax^2 + bx + c$. If $a > 0$, it opens up. If $a < 0$, it open downward. We can determine the vertex by completing the square.

$$f(x) = a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a}.$$

The vertex is $(-\frac{b}{2a}, c - \frac{b^2}{4a})$.

3. **Power functions** $f(x) = x^n$, where n is any real numbers. e.g. $f(x) = x^{1/2}$, $f(x) = x^{-5/6}$, $f(x) = x^{\sqrt{2}}$.

4. **Polynomials** $f(x) = a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$, where n is a non-negative integers, $a_n \neq 0$ and a_i are real numbers.

5. **Rational function** $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials.

6. **Piecewise defined function** e.g. $f(x) = \begin{cases} x^2+1, & x<0; \\ 2x+1, & x \geq 0. \end{cases}$ The functions is defined piece by piece.

7. **exponential function** Let $a > 0$. Then $f(x) = a^x$ is the exponential function to the base a.(Section 4.1)

Exercises

1. Suppose that $f(x) = x(x+1)$ and $g(x) = \frac{x}{x+1}$. Find
(i) $f(x+1)$, (ii) $f(x) - f(x-1)$, (iii) $f(f(x))$,
(iv) $f(g(x))$, (v) $g(f(x))$, (vi) $g(\frac{1}{x})$.

2. Identify the domain of the following functions.
(i) $f(x) = \frac{1}{(x+3)^2}$, (ii) $g(x) = \sqrt{t-2}$, (iii) $g(x) = \sqrt{(t-2)^2}$.

3. Let $f(x) = x^3$ and let $h \neq 0$. Simplify the expression $\frac{f(x+h)-f(x)}{h}$.

4. Paul has a fever and he is undergoing a fever-reducing medication. The temperature of his body is given by $T(t) = 37 + \frac{2}{t+1}$.
(i) Find the temperature after one hour.
(ii) When will his temperature become 37.5.

5. A cylindrical can is to have a capacity (volume) of 24π cm^3. The cost of the material used for the top and bottom of the can is 3 cents/cm2, and the cost of the material used for the curved side is 2 cents/cm^2. Express the cost of constructing the can as a function of its radius.

6. Find the x- and y-intercept and also the vertex of the quadratic function.
(i) $x^2 + 2x - 3$, (ii) $2x^2 - x - 1$, (iii) $-x^2 - 3x - 1$,

7. Solve x in the following equations.
(i) $4^{2x-1} = 16$, (ii) $4^x(\frac{1}{2})^{3x} = 8$, (iii) $10^{x^2-1} = 10^3$.