Lecture 10: Optimization problems

Suppose that $f(x)$ is a function defined on a closed bounded interval $[a, b]$. Then $f(x)$ will attain its *absolute* maximum and minimum value on $[a, b]$.

Let $f(x)$ be a function defined on an interval $[a, b]$. Then we say that

$f(c)$ is the *absolute maximum* of $f(x)$ on $[a, b]$ if $f(c) \geq f(x)$ for all x in $[a, b]$

$f(d)$ is the *absolute minimum* of $f(x)$ on $[a, b]$ if $f(d) \leq f(x)$ for all x in $[a, b]$.

Collectively, absolute maxima and minima are called *absolute extrema*. The point c are called the *absolute maximum point* of $f(x)$ on $[a, b]$ and the point d are called the *absolute minimum point* of $f(x)$ on $[a, b]$.

To find these extremal point, we need three steps.

(i) Use differentiation to find the critical points and hence the relative maximum and minimum.

(ii) Find the value of $f(a)$ and $f(b)$ of $f(x)$ as they may be the absolute extrema.

(iii) Compare all the values you find in (i) and (ii).

Example 1. Find the absolute maximum and absolute minimum of the function

$$f(x) = 2x^3 + 3x^2 - 12x - 7.$$

on the interval $-3 \leq x \leq 0$.

Step 1: $f'(x) = 6x^2 + 6x - 12 = 6(x - 1)(x + 2)$. Set $f'(x) = 0$, we obtain $x = 1$ or $x = -2$. As $x = 1$ is outside our interval $[-3, 0]$, we need only consider $x = -2$. As $f''(x) = 12x + 6$, $f''(-2) = -18 < 0$, -2 is a relative maximum by the second derivative test. $f(-2) = 13$.

Step 2: Compute the value of $f(x)$ at $x = -3$ and $x = 0$, we obtain $f(-3) = 2$ and $f(0) = -7$.

Step 3: Compare all the values you find, the absolute maximum of $f(x)$ in $[-3, 0]$ is 13 and it occurs when $x = -2$. The absolute minimum of $f(x)$ in $[-3, 0]$ is -7 and it occurs when $x = 0$.

Exercises.

1. Find the absolute maximum and minimum for the following functions over the prescribed intervals.

(i) $f(x) = 3x^5 - 5x^3 - 3$ over $-2 \leq x <\leq 0$,

(ii) $f(x) = 10x^6 + 24x^5 + 15x^4 + 3$, over $-1 \leq x <\leq 1$,

(iii) $f(x) = \frac{x+6}{x^2-x-6}$, over $-2 \leq x \leq 3$.

(iv) $f(x) = \frac{x^2}{x+1}$ over $-1/2 \leq x \leq 1$

(v) $f(x) = x + \frac{1}{x^2}$, over $x > 0$.

2. The highway department is planning to build a picnic park for motorists along a major highway. The park is to be rectangular with an area of 5000 m^2 and is to be fenced off on the three sides not adjacent to the highway. What is the least amount of fencing required for this job? How long and wide should the park be for the fencing to be minimized?

3. A cylindrical container is to be constructed to hold 100 cm^3 of loose tea leaves. The cost of the material used for the top and bottom of the container is 3 cents/cm^2, and the cost of the material used for the curved side is 2 cents/cm^2. Use calculus to find the radius and height of the container that is the least costly to construct.

4. A cable is to be run from a power plant on one side of a river that is 900 m wide to a factory on the other side, 3000 m downstream. The cost of running the cable under the water is $5 per metre, while the cost over land is $4 per metre. What is the most economical route over which to run the cable?

5. A manufacturer can produce souvenir T-shirts at a cost of $4 each. The shirts have been selling for $10 each, and at this price, tourists have been buying 4000 shirts a month. The manufacturer is planning to raise the price of the shirts and estimates that for each $1 increase in the price, 400 fewer shirts will be sold each month. At what price should the manufacturer sell the shirts to maximize profit?

6. A triangle is positioned with its hypotenuse on a diameter of a circle. If the circle has radius 4, what are the dimensions of the triangle of greatest area?