Lecture 2: Limit of functions

1. Limit. Let $f(x)$ be a function and let c be a number. We say that

$$\lim_{x\to c} f(x) = L$$

if $f(x)$ approaches to L whenever x approaches to c.

An easy example is that if $f(x) = x$. Then $\lim_{x\to c} x = c$ by observing directly from the graph. We consider another two functions.

$$f(x) = \frac{x^2-9}{x-3},\ g(x) = x+3,\ h(x) = \begin{cases} x+3, & x \neq 3; \\ 7, & x = 3. \end{cases}$$

The domain of $f(x)$ is all real numbers except 3, so $f(x)$ is undefined at $x = 3$. $f(x)$, $g(x)$ and $h(x)$ are different functions. However,

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x+3)(x-3)}{x-3} = \lim_{x\to 3}(x+3) = 6.$$

$$\lim_{x\to 3} h(x) = \lim_{x\to 3}(x+3) = 6.$$

When finding the limit, we are considering the behavior of the functions *around* the points, not the value of $f(x)$ on that point.

Limit of functions may fail to exist. For example,

$$\lim_{x\to 2} \frac{x+1}{x-2}.$$

As x goes to 2, $x+1$ goes to 3, but the denominator goes to 0. Therefore, $\frac{x+1}{x-2}$ goes to 3/0, not a number.

2. Two-sided limit. If the function is piecewisely defined, we will write $\lim_{x\to c^-} f(x)$ to be the limit of $f(x)$ when x approaches c from the *left* hand side of c (i.e. $x < c$) and $\lim_{x\to c^+} f(x)$ to be the limit of $f(x)$ when x approaches c from the *right* hand side of c (i.e. $x < c$).

$$f(x) = \begin{cases} 1-x^2, & x \leq 2; \\ 2x+1, & x > 2. \end{cases}$$

Then $\lim_{x\to 2^-} f(x) = 1 - 2^2 = -3$ and $\lim_{x\to 2^-} f(x) = 2(2)+1 = 5$. $\lim_{x\to 2} f(x)$ does not exist. In general,
$\lim_{x\to c} f(x)$ *exists if and only if* $\lim_{x\to c^-} f(x)$ *and* $\lim_{x\to c^+} f(x)$ *both exists and equal.*

3. Manipulation of limits. Some basic rules: Assume that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, then

(i) $\lim_{x\to c}(f(x) \pm g(x)) = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$.
(ii) $\lim_{x\to c}(f(x)g(x)) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$.
(iii) $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$.

Other techniques: There are many many techniques, basic two are factorization and rationalization

(*Factorization*) $\lim_{x\to 5}\frac{x^2-3x-10}{x-5}=\lim_{x\to 5}\frac{(x-5)(x+2)}{x-5}=\lim_{x\to 5}(x+2)=7.$

(*Rationalization*)

$$\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}=\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}=\lim_{x\to 1}\frac{x-1}{(x-1)(\sqrt{x}+1)}=\lim_{x\to 1}\frac{1}{\sqrt{x}+1}=\frac{1}{2}.$$

4. Limit at infinity. We write

$$\lim_{x\to+\infty}f(x)=L$$

if $f(x)$ approaches L as x tends to positive infinity. Similarly,

$$\lim_{x\to-\infty}f(x)=L$$

if $f(x)$ approaches L as x tends to negative infinity.

This following is an important property in evaluating limits at infinity. For any $k>0$,

$$\lim_{x\to\pm\infty}\frac{1}{x^k}=0$$

For example,

$$\lim_{x\to+\infty}\frac{x^2+2x+3}{2x^2-x}=\lim_{x\to+\infty}\frac{1+\frac{2}{x}+\frac{3}{x^2}}{2-\frac{1}{x}}=\frac{1+0+0}{2-0}=\frac{1}{2}.$$

5. Continuous functions We say that $f(x)$ is continuous at $x=c$ if

$$\lim_{x\to\infty}f(x)=f(c).$$

The function $h(x)=\begin{cases}x+3, & x\neq 3;\\ 7, & x=3.\end{cases}$ is not continuous at $x=3$. $f(x)=\frac{x+1}{x-2}$ is not continuous at $x=2$ since $f(x)$ is undefined at $x=2$. However, if we define

$$F(x)=\begin{cases}\frac{x^2-9}{x-3}, & x\neq 3;\\ 6, & x=3.\end{cases}$$

Then $\lim_{x\to 3}F(x)=6=F(3)$, so F is continuous at $x=3$.