Lecture 1: Differentiations

Let $f(x)$ be a function. The *average rate of change* of $f(x)$ from $x = a$ to $x = b$ is defined to be

$$\text{Rate}_{\text{ave}} = \frac{f(b) - f(a)}{b - a}.$$

Our interest is to define a notion of *instantaneous rate of change*. Note that the average rate of change of $f(x)$ from x to $x + h$ is given by

$$\frac{f(x+h) - f(x)}{(x+h) - h} = \frac{f(x+h) - f(x)}{h}.$$

The derivative of $y = f(x)$ at x is defined to be

$$f'(x) = \frac{d}{dx} f(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

A function $f(x)$ is said to be *differentiable* at x if the limit above exists. The process of finding derivatives given a function $f(x)$ is called *differentiation*.

Two interpretations of the $f'(x)$

(1) It is the *slope* of the straight line that touches the $f(x)$ at the point x. This line is called the *tangent line*.

(2) It is the *instantaneous rate of change* of $f(x)$ with respect to x. For example, if $f(t)$ is the distance traveled by a car at time t, $f'(t)$ is the speed of the car at time t.

One can also find the higher order derivatives by applying the differentiation to $f'(x)$. i.e.

$$f''(x) = \frac{d^2 y}{dx^2} = \frac{d}{dx} f'(x).$$

Geometrically, if $f'(x) > 0$, then f is *increasing* at the point x. If $f'(x) < 0$, then $f(x)$ is *decreasing* at the point x.

Main important formulae:

1. $\frac{d}{dx} k = 0$, for any constant k.
2. $\frac{d}{dx}(mx + c) = m$.
3. $\frac{d}{dx}\left(af(x)\right) = a \frac{d}{dx} f(x)$.
4. $\frac{d}{dx}\left(f(x) \pm g(x)\right) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$.
5. $\frac{d}{dx}(x^n) = nx^{n-1}$, for any real numbers n.
6. $\frac{d}{dx}(e^x) = e^x$, where $e = 2.71828....$ is the constant such that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$ (Section 4.2).

Examples. $\frac{d}{dx} x^3 = 3x^2$,
$\frac{d}{dx} x^{-3} = -3x^{-4} = -\frac{3}{x^4}$,

$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$.

1. Find the equation of the tangent line for the function $f(x) = x^3 - 4x^2 + 4$ at the point $x = 1$.

Solution: $f'(x) = 3x^2 - 8x$. At $x = 1$, slope of tangent line $= f'(1) = 3 - 8 = -5$. Also the tangent line passes through the point $(1, f(1)) = (1, 1^3 - 41^2 + 4) = (1, 1)$. The equation of the tangent line is

$$y - 1 = -5(x - 1) \Longrightarrow y = -5x + 6.$$

2. Suppose the distance traveled by a car is $f(t) = 3t + 5t^2$ metres. Find the average speed of the car between $t = 0s$ to $t = 5s$. Find also the (instantaneous) speed of the car. What is the acceleration (the rate of change of the speed)?

Solution: Average speed over $t = 0$ to $t = 5 = \frac{f(5)-f(0)}{5-0} = \frac{15+125}{5} = 28$ m/s. The speed of the car $= f'(t) = 3 + 10t$ m/s, the acceleration of the car $= 10$ m/s^2.