Lecture 4: Three important rules in differentiation

In doing differentiation of more complicated functions, the following three important rules will be used repeatedly.

1. **Product Rule**

$$\frac{d}{dx}\left(f(x)g(x)\right) = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x).$$

2. **Quotient Rule**

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{(g(x))^2}.$$

3. **Chain Rule**

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x).$$

Several simple examples illustrating the use of these rules.

Example 1.

$$\begin{aligned}\frac{d}{dx}(2x+1)(4x+3) =& (2x+1)\frac{d}{dx}(4x+3) + (4x+3)\frac{d}{dx}(2x+1)\\ =& (2x+1)(4) + (4x+3)(2)\\ =& 16x + 10.\end{aligned}$$

Example 2.

$$\begin{aligned}\frac{d}{dx}\left(\frac{2x+1}{4x+3}\right) =& \frac{(4x+3)\frac{d}{dx}(2x+1) - (2x+1)\frac{d}{dx}(4x+3)}{(4x+3)^2}\\ =& \frac{(4x+3)(2) - (2x+1)(4)}{(4x+3)^2}\\ =& \frac{2}{(4x+3)^2}.\end{aligned}$$

Example 3.

$$\begin{aligned}\frac{d}{dx}(2x+1)^2 =& 2(2x+1)\frac{d}{dx}(2x+1)\\ =& 2(2x+1)2 = 4(2x+1),\end{aligned}$$

$f(u) = u^2$ and $g(x) = 2x+1$.

To master the three rules, you must practise a lot.

Exercises

1. Find the derivatives for the following functions.

(a) $(3-x)(x^2-1)$, (b) $\frac{1-x}{1+x}$, (c) $\frac{1}{1+x+x^2}$, (d) $\frac{2t}{(t+2)^2}$,

(e) $\frac{(2x-1)(x+3)}{x+1}$, (f) $\sqrt{s^2-3s+2}$, (g) $\sqrt{\frac{1-x}{1+x}}$, (h) $(2x+1)^3(x-1)^5$

2. Find the value of x of the following function for which the tangent line is horizontal.

(a) $(x^2+x)^2$, (b) $x^3(2x^2+x-3)^2$, (c) $\frac{2x+5}{(1-2x)^3}$, (d) $(x-1)^2(2x+1)^3$.

3. Find the equation of the tangent line at $x=1$ for the following function.

(a) $(9x-1)^{-1/3}$, (b) $\frac{1}{(2x-1)^6}$, (c) $x^2\sqrt{2x+3}$.

4. Find the second derivatives for the following functions.

(a) $\frac{2}{5t+1}$, (b) $(1-2x^3)^4$, (c) $\sqrt{1+x^2}$

5. The bacteria growth after an introduction of a toxin is given by

$$B(t) = \frac{24t+10}{t^2+1}$$

million t hours after the introduction.

(a) At what rate the number of bacteria is changing 1 hour after the toxin was introduced?

(b) At what time does the number of bacteria begin to decline?

(c) What happen to the number of bacteria in the long run?