Lecture 5: Implicit differentiation

A lot of time the independent variable x and the dependent variable y are defined by a equation. For example,

$$2x^3 + y^4 - xy = 1,\ x^2 + y^2 = 1,\ x^2 - 2xy^2 = x^3 + y^2 + 1.$$

If you plot the graph, they are not graph of function. However, these graph still have a well-defined tangent line and you can find out its slope using *implicit differentiation.*

The way to do this differentiation is to use chain rule. We differentiate both side with respect to x and

$$\frac{d}{dx} f(y) = \frac{d}{dy}(f(y))\frac{dy}{dx}$$

Example 1. Find $\frac{dy}{dx}$ for $x^2 + y^2 = 1$.

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1).$$
$$2x + \frac{d}{dx}(y^2) = 0.$$
$$2x + \frac{d}{dy}(y^2)\frac{dy}{dx} = 0.$$
$$2x + 2y\frac{dy}{dx} = 0.$$
$$\frac{dy}{dx} = \frac{-x}{y}.$$

Example 2. Find $\frac{dy}{dx}$ for $2x^3 + y^4 - xy = 1$.

$$\frac{d}{dx}(2x^3 + y^4 - xy) = \frac{d}{dx}(1).$$
$$6x^2 + 4y^3\frac{dy}{dx} - \frac{d}{dx}(xy) = 0.$$
$$6x^2 + 4y^3\frac{dy}{dx} - x\frac{dy}{dx} - y = 0.$$
$$(4y^3 - x)\frac{dy}{dx} = y - 6x^2.$$
$$\frac{dy}{dx} = \frac{y - 6x^2}{4y^3 - x}.$$

Exercises

1. Find $\frac{dy}{dx}$ for the following

(a) $x^2 + y = x^3 + y^2$, (b) $\frac{1}{x} + \frac{1}{y} = 1$, (c) $\sqrt{x} + \sqrt{y} = 1$,

(d) $x^2y^3 = xy + 1$, (e)$(x^2 - y)^2 = xy$.

2. Find the equation of the tangent line for the following graph at the specified point.

(a) $x^2 = y^3$ at $(8, 4)$.
(b) $\frac{1}{x} - \frac{1}{y} = 2$ at $(\frac{1}{4}, \frac{1}{2})$.
(c) $x^2y^3 - 2xy = 6x + y + 1$ at $(0, -1)$
(d) $(x^2 + 2y)^3 = 2xy^2 + 64$ at $(0, 2)$

3. Find the points on the graph so that the tangent line is horizontal or vertical.
(a) $x^2 + xy + y = 3$, (b) $\frac{y}{x} - \frac{x}{y} = 5$, (c) $x^2 - xy + y^2 = 3$.