Lecture 7: Curve sketching: First derivatives

We can describe the behavior of a function from the information about the first and second derivatives. In this note, we first discuss **first derivatives.**

$f(x)$ is said to be *increasing* on the interval $[a, b]$ if $f(x_1) \leq f(x_2)$ whenever $x_1 < x_2$.
$f(x)$ is said to be *decreasing* on the interval $[a, b]$ if $f(x_1) \geq f(x_2)$ whenever $x_1 < x_2$

Theorem 1.
(i) $f(x)$ is increasing on the interval $[a, b]$ if and only if $f'(x) > 0$ on $[a, b]$.
(ii) $f(x)$ is decreasing on the interval $[a, b]$ if and only if $f'(x) < 0$ on $[a, b]$.

c is called a *critical number* of $f(x)$ if $f'(c) = 0$, in this case, $(c, f(c))$ is called the *critical point* of $f(x)$. Note that in many textbooks, c is also referred as critical point.

The graph of the function $f(x)$ is said to have a *relative maximum* at $x = c$ if $f(c) \geq f(x)$ for all x in an interval $a < x < b$ containing c. Similarly, the graph has a *relative minimum* at $x = c$ if $f(c) \leq f(x)$ on such an interval. Collectively, the relative maxima and minima of f are called its *relative extrema.*

Theorem 2. *If c is a relative maximum or a relative minimum point of $f(x)$, then it must be a critical point of $f(x)$ (i.e. $f'(c) = 0$).*

Note that a critical point may be a relative maximum, a relative minimum or *a stationary point.* The following first derivative test gives us the way to determine the type of the critical point.

Theorem 3 (First derivative test). *Let c be a critical number for $f(x)$. Then the critical point $(c, f(c))$ is*
(i) a relative maximum if $f(x) > 0$ to the left of c and $f(x) < 0$ to the right of c.
(ii) a relative minimum if $f(x) < 0$ to the left of c and $f(x) > 0$ to the right of c.
(iii) not a relative extremum if $f(x)$ has the same sign on both sides of c.

Exercises.
1. Find the interval of increase and decrease of the following functions.
(i) $f(x) = x^3 + 3x^2 + 1$, (ii) $f(x) = x^5 - 5x^4 + 10$,
(iii) $f(x) = \frac{x}{(x+3)^2}$, (iv) $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$.

2. Find the critical point of the function and identify its type.
(i) $f(x) = 324x - 72x^2 + 4x^3$, (ii) $f(x) = 10x^6 + 24x^5 + 15x^4 + 3$,
(iii) $f(x) = (x - 1)^5$, (iv) $f(x) = \frac{x^2}{x^2+x-2}$, (v) $\frac{x^2}{x-1}$.

3. The concentration of a drug t hours after being injected into the arm of a patient is given by

$$C(t) = \frac{0.15t}{t^2 + 0.81}.$$

When does the maximum concentration occur?