Lecture 9: Curve sketching: Asymptotes

We say that the line $x = c$ is a *vertical asymptote* of the graph of f(x) if

$$\lim_{x\to c} f(x) = \infty \text{ (may be } +\infty \text{ or } -\infty).$$

In general, a rational function $\frac{p(x)}{q(x)}$ has a vertical asymptote $x = c$ whenever $q(c) = 0$ but $p(c) \neq 0$.

Example 1. Let $f(x) = \frac{x^2-9}{x(x-3)}$. Note that

$$\lim_{x\to 0} f(x) = \infty$$

as the denominator is zero when x approaches 0 but the numerator is non-zero. Therefore, $x = 0$ is a vertical asymptote. On the other hand,

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{(x+3)(x-3)}{x(x-3)} = \lim_{x\to 3} \frac{(x+3)}{x} = \frac{6}{3} = 2.$$

There is *no* vertical asymptote at $x = 3$.

The horizontal line $y = b$ is called a *horizontal asymptote* of the graph of $y = f(x)$ if

$$\lim_{x\to\pm\infty} f(x) = b.$$

Example 2. Let $f(x) = \frac{x^2-9}{x(x-3)}$. Note that

$$\lim_{x\to\infty} \frac{x^2-9}{x^2-3x} = 1.$$

There is a horizontal asymptote $y = 1$.

Exercises.

1. For the following functions, find vertical and horizontal asymptotes (if exists).
(i) $f(x) = \frac{x}{2-x}$, (ii) $f(x) = \frac{x^2-1}{x^2+1}$,
(iii) $f(x) = \frac{x^2+3x-5}{x^2-5x+6}$.

2. For the following functions, find
(i) the interval of increase/decrease and also the relative extrema.
(ii) the interval for which the function is convex/concave and also the point of inflection.
(iii) the vertical and horizontal asymptotes.
(iv) sketch the graph.
(I)$f(x) = x^3 + 3x^2 + 1$, (II) $f(x) = x^5 - 5x^4 + 10$,
(III) $f(x) = \frac{x}{(x+1)^2}$, (IV) $f(x) = \frac{2x}{x^2-1}$.

3. Find constants A and B so that the graph of the function

$$f(x) = \frac{Ax + 2}{8 - Bx}$$

has $x = 4$ as a vertical asymptote and $y = -1$ as a horizontal asymptote.

4. A manufacturer of motorcycles estimate that if x thousand dollars are spent on advertising, then

$$M(x) = 2300 + \frac{125}{x} - \frac{500}{x^2}$$

motorcycles will be sold.

(i) Sketch $M(x)$.

(ii) What level of advertising expenditure results in maximum sales?