1K03 E, Test 2 Solution.

1. D

$f'(x) = 3x^2$

slope of tangent line at $x=3$ $= f'(3) = 3(3)^2 = 27$

2. B

By Product Rule,

$\frac{dy}{dx} = (3-x^2) \cdot (3x^2) + (x^3+6)(-2x)$

$= 9x^2 - 3x^4 - 2x^4 - 12x$

$= -5x^4 + 9x^2 - 12x$

3. A

$f'(x) = \frac{(x^3-1) \cdot 0 - 1 \cdot (3x^2)}{(x^3-1)^2}$

$= \frac{-3x^2}{(x^3-1)^2}$

4. C

$f'(x) = \frac{1}{3}(9x^2-1)^{-2/3} \cdot \left(\frac{d}{dx}(9x^2-1)\right)$

$= \frac{1}{3}(9x^2-1)^{-2/3}(18x)$

$= 6x(9x^2-1)^{-2/3}$

$= \frac{6x}{(9x^2-1)^{2/3}}$

5. (B) $G'(x) = 3x^2 + 3$

$G''(x) = 6x$

6. (C) $\frac{d}{dx}(xy) = \frac{d}{dx}(1+y^2)$

$x\frac{dy}{dx} + y = 2y\frac{dy}{dx}$

$(x-2y)\frac{dy}{dx} = -y$

$\frac{dy}{dx} = \frac{-y}{x-2y}$

7. (D) $f'(t) = \frac{1}{2}(t^2+t+5)^{-\frac{1}{2}}(2t+1)$

$= \frac{2t+1}{2\sqrt{t^2+t+5}}$

The rate required $= f'(4) = \frac{2(4)+1}{2\sqrt{4^2+4+5}} = \frac{9}{2\sqrt{25}} = \frac{9}{10}$

$= 0.9$

8. (E) $f'(x) = 2(x^2-4)(2x)$

$= 4x(x^2-4)$

$= 4x^3 - 16x$

$f''(x) = 12x^2 - 16$

$f'''(x) = 24x$

$f^{(4)}(x) = 24$

$f^{(5)}(x) \equiv 0$

$\therefore n = 5.$

Part 2: From questions 9 to 11. Please write down your solutions with steps in detail in the space provided below the questions.

9. (4 points) Let

$$f(x) = \frac{x}{x^2+1}.$$

Find all values of x such that the tangent line is horizontal.

$$f'(x) = \frac{(x^2+1)\cdot\frac{d}{dx}(x) - x\,\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$ 1

$$= \frac{(x^2+1)\cdot 1 - x(2x)}{(x^2+1)^2}$$ 0.5

$$= \frac{x^2+1-2x^2}{(x^2+1)^2}$$

$$= \frac{1-x^2}{(x^2+1)^2}.$$ 0.5

If the tangent line is horizontal at x, then 1

$$f'(x) = 0$$

$$\frac{1-x^2}{(x^2+1)^2} = 0$$

$$1-x^2 = 0$$

$$(1+x)(1-x) = 0$$

$$x = 1 \text{ or } -1.$$

0.5 0.5

10 (a). (2 points) State the definition of the derivative of a function $f(x)$ (in the form of limit).

(b). (2 points) Write down TWO interpretations of the derivatives.

a). $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

b. (i). $f'(x)$ is the slope of the tangent line to $y = f(x)$ at the point x.

(ii) $f'(x)$ is the instantaneous rate of change of the function $f(x)$ at the point x.

11. Consider

$$y = \sqrt{25 - x^2}. \quad = (25 - x^2)^{\frac{1}{2}}$$

(a). (2 points) Find $\frac{dy}{dx}$.
(b). (2 points) Find the equation of the tangent line at $x = 3$.
(c). (2 points) Find the equation of the tangent line at $x = -4$.
(d). (2 points) Are the tangent lines in (b) and (c) perpendicular to each other? Explain.

a)

$$\frac{dy}{dx} = \frac{1}{2}(25 - x^2)^{-\frac{1}{2}} \frac{d}{dx}(25 - x^2) \qquad 1$$

$$= \frac{1}{2} \frac{1}{\sqrt{25 - x^2}} \cdot (-2x)$$

$$= \frac{-x}{\sqrt{25 - x^2}} \qquad 1$$

b) Slope of the tangent line at $x = 3$ $= \left.\frac{dy}{dx}\right|_{x=3}$

$$= \frac{-3}{\sqrt{25 - 3^2}}$$

$$= \frac{-3}{4}.$$

The tangent passes through the point $= (3, \sqrt{25 - 3^2})$

$$= (3, 4)$$

Eqt:

$$y - 4 = \frac{-3}{4}(x - 3)$$

$$y = -\frac{3}{4}x + \frac{9}{4} + 4$$

$$y = -\frac{3}{4}x + \frac{25}{4}. \qquad 2$$

11. Consider

$$y = \sqrt{25 - x^2}.$$

(a). (2 points) Find $\frac{dy}{dx}$.
(b). (2 points) Find the equation of the tangent line at $x = 3$.
(c). (2 points) Find the equation of the tangent line at $x = -4$.
(d). (2 points) Are the tangent lines in (b) and (c) perpendicular to each other? Explain.

c) Slope of the tangent line at $x=-4$ $= \dfrac{-(-4)}{\sqrt{25-(-4)^2}}$

$$= \frac{4}{\sqrt{25-16}}$$

$$= \frac{4}{3}$$

The tangent line passes through the point $= (-4, \sqrt{25-(-4)^2})$

$$= (-4, 3)$$

Eqt: $y - 3 = \frac{4}{3}(x-(-4))$

$$y = \frac{4}{3}x + \frac{16}{3} + 3$$

$$y = \frac{4}{3}x + \frac{25}{3}$$

2

d) Since the (slope of the tangent in (b)) $\times$ (slope of the tangent in (c))

$$= -\frac{3}{4} \cdot \frac{4}{3} = -1$$

1.5

The tangent lines are perpendicular. 0.5