1. Let $y^2 = 2x^3$. Find $\frac{dy}{dx}$.

A. $\frac{3x^2}{2y^2}$

B. $\frac{3x^2}{2y}$

C. $\frac{3x^2}{y}$

D. $\frac{3x^3}{y}$

E. $\frac{3x^3}{2y^2}$

$2y\frac{dy}{dx} = 6x^2$

$\frac{dy}{dx} = \frac{6x^2}{2y} = \frac{3x^2}{y}$.

2. Let $y^2 = x^3 - 4x + 1$. Find the slope of the tangent line at the point $(-2, 1)$.

A. 0

B. 1

C. 2

D. 3

E. 4

$2y\frac{dy}{dx} = 3x^2 - 4$

$\frac{dy}{dx} = \frac{3x^2-4}{2y}$

At $(-2,1)$ slope of tangent $= \frac{3(-2)^2-4}{2(1)} = \frac{8}{2} = 4$

3. Suppose that a spherical balloon is releasing its air inside at a rate of $3\pi\ cm^3$ per second. What is the rate the radius of the balloon is decreasing when the balloon has a radius of 1 cm?

(Hint: The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$)

A. 0.25 cm/s

B. 0.5 cm/s

C. 0.75 cm/s

D. 1 cm/s

E. None of the above.

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

When $\frac{dV}{dt} = -3\pi$, $r = 1$

$-3\pi = 4\pi(1)\cdot\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = -0.75$

4. Let $g(x) = x^2 - x + 1$. Find the x-coordinates of the vertex of $g(x)$.

A. $\frac{1}{4}$

B. $\frac{1}{2}$

C. 1

D. $-\frac{1}{2}$

E. $-\frac{1}{4}$

$g'(x) = 2x - 1$

At vertex, $g'(x) = 0$

$2x - 1 = 0$

$x = \frac{1}{2}$

5. Let
$$f(x) = x^6 - x^5 + 100.$$
Find the interval for which $f(x)$ is concave (concave down).

A. $0 < x < \frac{2}{3}$

B. $x < 0$ or $x > \frac{2}{3}$

C. $x < \frac{5}{6}$

D. $x > \frac{5}{6}$

E. $0 < x < \frac{5}{6}$

$f'(x) = 6x^5 - 5x^4$

$f''(x) = 30x^4 - 20x^3$

$= 10x^3(3x-2)$

	$x<0$	$x=0$	$0<x<\frac{2}{3}$	$x=\frac{2}{3}$	$x>\frac{2}{3}$
f''	$+$	0	$-$	0	$+$

6. Let $f(x) = \frac{x^2}{x^2-4}$. Find the interval for which the function is increasing.

A. $x > 0$

B. $x < 0$

C. $-2 < x < 2$

D. $0 < x < 2$

E. $-2 < x < 0$

$f'(x) = \frac{(x^2-4)(2x) - x^2(2x)}{(x^2-4)^2}$

$= \frac{-8x}{(x^2-4)^2}$

$f'(x) > 0 \iff -8x > 0 \quad (\because (x^2-4)^2 > 0)$

$\iff x < 0$

7. Let
$$f(x) = \frac{x^2}{x^2 + 2x - 15}.$$
How many vertical asymptotes does $f(x)$ have?

A. 0

B. 1

C. 2

D. 3

E. 4

$x^2 + 2x - 15 = (x-3)(x+5)$

As ~~putting~~ putting 3 and 5 to numerator are non-zero.

$x=3$, $x=5$ are vertical asymptotes.

8. Let
$$f(x) = \frac{3\sqrt{x}}{\sqrt{x}+2}.$$
Find the horizontal asymptotes of $f(x)$ as x tends to infinity.

A. $y = 0$

B. $x = 0$

C. $x + y = 3$

D. $y = 3$

E. $x = 3$

$\lim_{x\to\infty} \frac{3\sqrt{x}}{\sqrt{x}+2} = \lim_{x\to\infty} \frac{3}{1+\frac{2}{\sqrt{x}}} = \frac{3}{1+0} = 3$

$\therefore$ $y=3$ is the horizontal asymptote.

From questions 9 to 11. Please write down your solutions with steps in detail in the space provided.

9. Let
$$f(x) = x^3 - 9x^2 + 24x.$$
(i) (2 points) Find $f'(x)$ and $f''(x)$.
(ii) (2 points) Find the relative maximum and relative minimum of $f(x)$.
(iii) (2 points) Find the point of inflection of $f(x)$.
(iv) (2 points) Sketch $f(x)$. Indicate the points you found in (ii) and (iii).

(i) $f'(x) = 3x^2 - 18x + 24$

$f''(x) = 6x - 18$

(ii). Set $f'(x) = 0$

$3(x^2 - 6x + 8) = 0$

$3(x-2)(x-4) = 0$

Therefore $x = 2$ or $x = 4$.

	$x<2$	$x=2$	$2<x<4$	$x=4$	$x>4$
f'	$+$	0	$-$	0	$+$

OR Note that $f'(x) > 0$

$\Leftrightarrow (x-2)(x-4) > 0$

$\Leftrightarrow x<2$ or $x>4$.

Similarly $f'(x) < 0 \Leftrightarrow 2 < x < 4$.

As $f(2) = 20$ $f(4) = 16$

$\therefore$ $(2, 20)$ is a relative maximum,

$(4, 16)$ is a relative minimum.

(iii) Set $f''(x) = 0$

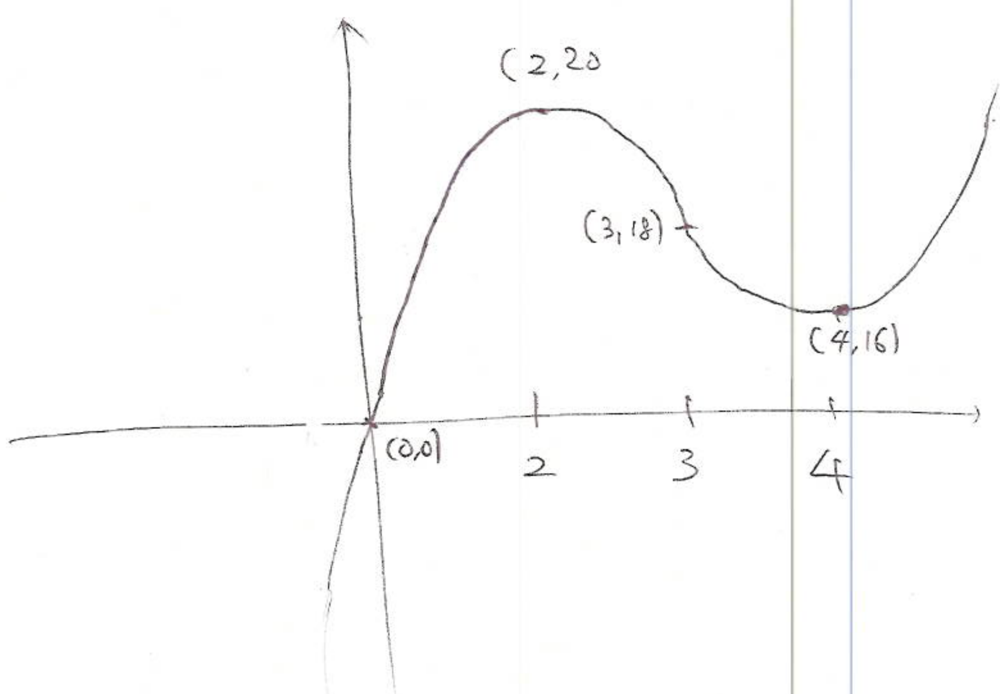
$$6(x-3) = 0$$
$$x = 3$$

	$x<3$	$x=3$	$x>4$
f''	$-$	0	$+$

$$f''(x) > 0 \iff 6(x-3) > 0 \iff x > 3$$
$$f''(x) < 0 \iff 6(x-3) < 0 \iff x < 3.$$

Hence, (as $f(3) = 18$), $(3, 18)$ is a point of inflection.

iv



10. (3 points) Let

$$(x^2 + y^2)^2 = 2xy.$$

Find $\frac{dy}{dx}$

$$\frac{d}{dx}\left(x^2+y^2\right)^2 = \frac{d}{dx}(2xy)$$

Chain Rule

Product Rule

$$2(x^2+y^2)\,\frac{d}{dx}\left(x^2+y^2\right) = 2\left[x\frac{dy}{dx}+y\right]$$

$$2(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right) = 2x\frac{dy}{dx}+y$$

$$4x(x^2+y^2)+4y(x^2+y^2)\frac{dy}{dx} = 2x\frac{dy}{dx}+2y$$

$$\left[4y(x^2+y^2)-2x\right]\frac{dy}{dx} = 2y-4x(x^2+y^2)$$

$$\frac{dy}{dx} = \frac{2y-4x(x^2+y^2)}{4y(x^2+y^2)-2x}$$

$$= \frac{y-2x(x^2+y^2)}{2y(x^2+y^2)-x}$$

11. (5 points) An ice block used in a cooler at a campsite is modelled as a cube of side length s. The block currently has volume 125 000 cm^3 and is melting at a rate of 1000 cm^3 per hour.

(a) At what rate the side length is changing currently?

(b) What is the rate of change of the total surface area?

a) $V = s^3.$

$$\frac{dV}{dt} = 3s^2 \cdot \frac{ds}{dt}$$

Currently, the side length $= \sqrt[3]{125000} = 50$ cm.

and $\frac{dV}{dt} = -1000$ cm^3/h.

$$-1000 = 3(50)^2 \cdot \frac{ds}{dt}$$

$$\frac{ds}{dt} = \frac{-1000}{7500} = -\frac{2}{15}.$$

$\therefore$ The side length is ~~at~~ decreasing a rate of $\frac{2}{15}$ cm/h.

b. Let A = Total surface area

$$A = 6s^2$$

$$\frac{dA}{dt} = 12s\,\frac{ds}{dt}$$

$$\therefore \frac{dA}{dt} = 12(50)\cdot\left(\frac{-2}{15}\right) = 80 \text{ cm}^2/\text{h}.$$

$\therefore$ Total surface area is decreasing at rate of 80 cm^2/h.