Notes on Linear Algebra

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In this class linear algebra plays a key role, a role which should become a little more clear as the course progresses. These notes are meant to give an introduction to those concepts, as well as examples from linear algebra that are directly related to the material in this course. Furthermore, these notes follow very closely to the lectures given on June 21 and June 23, 2012.

1 Vector Spaces

In Section 7.6 of the textbook [2] the authors define a vector space (Definition 7.6.1) by specifying certain *axioms* that it needs to satisfy. Many of the details of the definition are not very relevant to us now and, since you can find it in the text, I won't reproduce the definition here. When it comes down to it, the axiomatic definition is just a way of formalizing the following 'intuitive definition': A **vector space** is a set V in which it makes sense to

- add any two elements in V (called **vector addition** or simply **addition**), and
- multiply any element of V by real numbers (called **scalar multiplication**).

Here is some additional terminology: the elements of V are called **vectors** and the real numbers are called **scalars**. In this class we will use the word *space* to refer to a set that is also a vector space. If we only use the word *set* then we are do not have any vector space-type structure in mind.

The next two subsections are very important in that every example we will see in this class will come from one of these in a sense we will make precise in the next section.

1.1 The Vector Space $\mathbb{R}^n$

Define the following set

$$\mathbb{R}^n := \{\langle x_1, \ldots, x_n \rangle \mid x_j \in \mathbb{R} \text{ for } j = 1, \ldots, n\}.$$

The notation '$\mathbb{R}^n$' is read 'aar-en'. We can give $\mathbb{R}^n$ the structure of a vector space by defining addition, scalar multiplication and the zero vector as follows: Given $\langle x_1, \ldots, x_n \rangle, \langle y_1, \ldots, y_n \rangle \in \mathbb{R}^n$ and $k \in \mathbb{R}$, we define

- **Addition in $\mathbb{R}^n$:** $\langle x_1, \ldots, x_n \rangle + \langle y_1, \ldots, y_n \rangle := \langle x_1 + y_1, \ldots, x_n + x_n \rangle$,
- **Scalar Multiplication in $\mathbb{R}^n$:** $k \cdot \langle x_1, \ldots, x_n \rangle := \langle kx_1, \ldots, kx_n \rangle$.
- **Zero in $\mathbb{R}^n$:** The zero in $\mathbb{R}^n$ is the element with zero in every component: $\langle 0, 0, \ldots, 0 \rangle$.

Said succinctly *addition and scalar multiplication are defined component-wise.* Then this way of defining addition, scalar multiplication, and the zero satisfies the axioms of a vector space from the book.

Exercise 1. *Verify that, with these definitions, $\mathbb{R}^n$ satisfies the axioms in the definition of vector space.*

> **Remark.** It is sometimes useful to use different notation to represent vectors in $\mathbb{R}^n$. For example, some authors use $(x_1, \ldots, x_n)$ rather than $\langle x_1, \ldots, x_n \rangle$. However, since parentheses are so overused in math, in this class I will primarily use the square brackets, $\langle\ \rangle$, instead.
>
> Sometimes it is also useful to think of the vectors in $\mathbb{R}^n$ as matrices with one column and n rows:
>
> $$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$
>
> This viewpoint is useful when talking about matrix multiplication, and we will adopt this notation a few times later when we discuss *linear transformations.*

1.1.1 The line $\mathbb{R}^1$

As a visual tool, we often think of $\mathbb{R}^1$ as the real number line. This identification is made by simply placing the vector $\langle x \rangle$ at the location x on the number line.

Note also that the following two sets are obviously very similar:

$$\mathbb{R} = \{x \mid x \in \mathbb{R}\}$$

$$\mathbb{R}^1 = \{\langle x \rangle \mid x \in \mathbb{R}\}$$

Some authors identify these two, and that is fine for most situations. However, there are a few places where this can lead to confusion, so when talking about the *vector space* I will use $\mathbb{R}^1$ and write its elements with square brackets $\langle x \rangle$; on the other hand, when talking about the set of *scalars* I will use $\mathbb{R}$ and not use any brackets for the elements. For example, $\langle 0 \rangle$ denotes the zero *vector* in $\mathbb{R}^1$, while 0 denotes the zero scalar, so

$$0 \langle x \rangle = \langle 0 \rangle$$

for all $\langle x \rangle \in \mathbb{R}^1$.

1.1.2 The plane $\mathbb{R}^2$

We can visualize $\mathbb{R}^2$ as the set of points in the plane. Explicitly, identify the vector $\langle a, b\rangle \in \mathbb{R}^2$ with the point whose x-coordinate is a and whose y-coordinate is b. Similar constructions can be done with $\mathbb{R}^n$ for any positive integer n.

1.2 The Vector Space $F(a, b)$

Define

$$F(0,1) := \{\text{real-valued functions } f \text{ defined on } [0,1]\}$$

More succinctly, write

$$F(0,1) = \{f : [0,1] \to \mathbb{R}\}$$

The 'F' stands for 'function'.

The following functions are elements in $F(0,1)$:

$$f(x) = x^2, \qquad g(x) = \sin(x), \qquad h(x) = \frac{1}{x-2}.$$

However, the function

$$i(x) = \frac{1}{x - 1/2}$$

is not in $F(0,1)$ because it is not defined at $1/2$. (Why is $h(x) = \frac{1}{x-2}$ in $F(0,1)$?)

Exercise 2. *Which of the following functions are in $F(0,1)$?*

$$\tan(x), \qquad e^x, \qquad |x|, \qquad \ln(x)$$

The 0 and 1 in the definition of $F(0,1)$ are not special. Indeed, for any real numbers a, b with $a < b$ we can define[1]

$$F(a,b) := \{f : [a,b] \to \mathbb{R}\}.$$

It will be useful to also consider functions defined on the whole real line. So we define

$$F(-\infty,\infty) := \{f : (-\infty,\infty) \to \mathbb{R}\}.$$

For simplicity, we will use the notation $F(a,b)$ to denote

We can turn $F(a,b)$ into a vector space by defining addition, scalar multiplication and the zero vector as follows: Let $f, g \in F(a,b)$ and $k \in \mathbb{R}$, and define

[1]The notation $F(a,b)$ is not standard. That is, if you are talking to someone outside of this class they will probably not know what $F(a,b)$ means (unlike $\mathbb{R}^n$ or $C(a,b)$, which are standard).

- **Addition in $F(a, b)$:** $f + g$ is the function whose value at $x \in [a, b]$ is $f(x) + g(x)$,
- **Scalar Multiplication in $F(a, b)$:** kf is the function whose value at $x \in [a, b]$ is $kf(x)$.
- **Zero in $F(a, b)$:** The 'zero element' of $F(a, b)$ is the function that is constantly zero. We denote this by 0_{fun}, and it is defined by

  $$0_{\text{fun}}(x) = 0$$

  for all $x \in [a, b]$.

For example, if f and g are the functions given by the formulas $f(x) = x^2$ and $g(x) = \sin(x)$ then $2f + 3g$ is the function given by the formula $2x^2 + 3\sin(x)$.

Exercise 3. *Verify that, with these definitions, $F(a, b)$ satisfies the axioms in the definition of vector space.*

Exercise 4. *Sketch the graph of $0_{fun} \in F(-\infty, \infty)$. Sketch the graph of $0_{fun} \in F(0, 1)$. How do these graphs differ?*

Note that the vectors in $F(a, b)$ are *functions*. Said differently, the *vectors* in this space happen to also be *functions*. This is in contrast to the vectors in $\mathbb{R}^2$, for example, which are ordered pairs of real numbers. Hence $F(a, b)$ is often called a **function space**.

2 Subspaces

In the previous section I mentioned that, in this course, all of our examples of vector spaces will come from $\mathbb{R}^n$ or $F(a, b)$ in some way. More precisely, all the other vector spaces will be *subspaces* of $\mathbb{R}^n$ or $F(a, b)$, (for suitably chosen n or a, b). Simply put, a subspace is a vector space that lies inside of another vector space. Here is the formal definition, which is what you need to refer to when proving that subsets are subspaces:

Definition 1. *Let V be a vector space and $W \subseteq V$ a subset*[2]*. Then we say that W is a* **subspace** *of V if each of the following conditions hold:*

1. *If $u, v \in W$ then $u + v \in W$;*
2. *If $u \in W$ and $k \in \mathbb{R}$ then $ku \in W$.*

The following theorem ties this definition in with the intuitive description of a subspace from the opening paragraph of this section:

[2] Given sets A, B, we say that A is a **subset** of B if every element of A is also an element of B. If this is the case, then we write $A \subseteq B$. Note that this definition *does* allow for the case where $A = B$. The little line at the bottom of symbol '$\subseteq$' is there to remind us of this (just like the line at the bottom of '$\leq$').

Theorem 2. *If V is a vector space and $W \subseteq V$ is a subspace, then W is a vector space. The addition and scalar multiplication are those inherited from V.*

Though this theorem isn't difficult to prove, I will not give a proof here. (Our textbook actually takes this as the *definition* of subspace.) However, one ingredient of the proof Theorem 2 is the first part of the following exercise. It provides an easy criteria for determining when a subset is *not* a subspace, as parts (a) and (b) illustrate.

Exercise 5. *Show that if $W \subseteq V$ is a subspace of V, then $0 \in W$. Use this to show that the following subsets of $\mathbb{R}^2$ are not subspaces:*

(a) $\{\langle x, y\rangle \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

(b) $\{\langle x, y\rangle \in \mathbb{R}^2 \mid x + 2y = 4\}$

In the converse direction, Theorem 2 can be used to prove certain sets *are* vector spaces, without having to verify all of the 10 axioms in the definition of vector space. This is illustrated in Corollary 4 below and through examples in Subsections 2.1 and 2.2.

Before proceeding to specific examples, we discuss two subspaces that appear in every vector space. These are defined in the statement of the next theorem.

Exercise 6. *This exercise is especially designed for those who are uncomfortable with proving things. The exercise is to spend some time going over the proof of this theorem. In particular, you should*

(a) Make sure you understand every step.

(b) After you have completed Part (a), try to understand the proof as a whole. For example, you could ask yourself, why are these ideas presented in this order?

Theorem 3 (The Trivial Subspace)**.** *Let V be a vector space. Then the following are subspaces of V*

$\{0\} \subseteq V$*, the subset containing only the zero vector;*

$V \subseteq V$*, the set V itself.*

These subspaces are called the **trivial** *subspaces.*

Proof. I will prove that $\{0\}$ is a subspace, and leave the proof that V is a subspace as an exercise. According to the definition of subspace, it suffices to verify each of the following claims:

Claim 1: If $u, v \in \{0\}$, then $u + v \in \{0\}$.

Claim 2: If $u \in \{0\}$ and $k \in \mathbb{R}$, then $ku \in \{0\}$.

To prove Claim 1, first observe that if $u, v \in \{0\}$, then $u = v = 0$ since $\{0\}$ only contains one element and this element is 0. Therefore

$$u + v = 0 + 0 = 0 \in \{0\},$$

which proves Claim 1.

The proof of Claim 2 is similar. If $k \in \mathbb{R}$ and $u \in \{0\}$, then we must have $u = 0$, and so

$$ku = k0 = 0 \in \{0\}.$$

$\square$

Exercise 7. *Complete the proof of Theorem 3 by proving that V is a subspace of itself.*

The following corollary is a cute application of the two theorems we have seen so far.

Corollary 4. *The set consisting of exactly one point $\{p\}$ is a vector space.*

Proof. Let V be any vector space ($\mathbb{R}^2$, for example). Then think of $\{p\}$ as a subset of V by identifying p with $0 \in V$. By Theorem 3, $\{p\}$ is a subspace of V, and so by Theorem 2, $\{p\}$ itself is a vector space. $\square$

2.1 Subspaces of $\mathbb{R}^n$

2.1.1 Classification of the Subspaces of $\mathbb{R}^1$

The only subspaces of $\mathbb{R}^1$ are the trivial subspaces: $\{\langle 0 \rangle\}$ and $\mathbb{R}^1$. To see this, suppose $W \subseteq \mathbb{R}^1$ is a subspace. We will show that if W is not the zero subspace $\{\langle 0 \rangle\}$ then it must be that $W = \mathbb{R}^1$. Toward this end, suppose $W \neq \{\langle 0 \rangle\}$. The only way this can happen is if there is some vector $\langle w \rangle \in W$ where $\langle w \rangle \neq \langle 0 \rangle$. Since W is a subspace, we must have

$$k \langle w \rangle \in W$$

for every $k \in \mathbb{R}$. I claim this implies that W contains every element of $\mathbb{R}^1$, and so $W = \mathbb{R}^1$. To prove the claim, let $\langle a \rangle \in \mathbb{R}^1$. Then take $k = a/w$ (this is allowed because $w \neq 0$) to get

$$\langle a \rangle = \langle (a/w)w \rangle = \langle kw \rangle = k \langle w \rangle \in W,$$

which proves the claim, since $\langle a \rangle \in \mathbb{R}^1$ was arbitrary (and hence this same argument works for any $\langle a \rangle$ you choose).

2.1.2 Classification of the Subspaces of $\mathbb{R}^2$

Apart from the trivial subspace, $\{\langle 0, 0 \rangle\}$ and $\mathbb{R}^2$, the only other subspaces of $\mathbb{R}^2$ are the lines that pass through the origin. The proof is not hard, but it is perhaps more instructive to think about *why* this is the case. For example, you might come up with a sub*set* that isn't one of the ones listed here, and try to figure out why it isn't a sub*space*.

2.1.3 Classification of the Subspaces of $\mathbb{R}^3$

Every subspace of $\mathbb{R}^3$ is either one of the trivial subspaces, a line through the origin, or a plane through the origin. Again, try to figure out why this is the case.

2.1.4 Subspaces are Flat

As the illustrated in $\mathbb{R}^1, \mathbb{R}^2$ and $\mathbb{R}^3$, the subspaces of $\mathbb{R}^n$ are flat: they look like lines, planes, and higher dimensional versions of these. This is a general phenomenon, and you should think of all subspaces (and all vector spaces for that matter) as being flat in some sense.

2.2 Subspaces of $F(a, b)$

2.2.1 Two Subspaces of Polynomials: P_n and P

Let $n \geq 0$ be an integer, and define

$$P_n := \{\text{polynomials of degree} \leq n\}$$

So we have

$$x^2 \in P_2, \qquad x^{15} + x^8 - 3 \in P_{1012},$$

but x^4 is not in P_3, and $\sin(x)$ is not in P_n for any n.

Since every polynomial $p(x)$ can be thought of as a function $p : (-\infty, \infty) \to \mathbb{R}$, we automatically have that P_n is a subset of $F(-\infty, \infty)$:

$$P_n \subseteq F(-\infty, \infty).$$

In fact, P_n is a sub*space*. Indeed, if p, q are polynomials of degree less than or equal to n, then $p + q$ is also a polynomial of degree less than or equal to n.

Exercise 8. *Convince yourself that this statement is true when $n = 10$: Write down two polynomials of degree ≤ 10. Add them together and verify that you again get a polynomial of degree ≤ 10. Try this a few more times. Can you come up with a way to prove this statement for general n (and not just $n = 10$)?*

Similarly, if you multiply p by a real number you get another polynomial, and this polynomial has degree less than or equal to n, since this was true of p. This proves that P_n is a subspace of $F(-\infty, \infty)$.

Now define

$$P := \{\text{polynomials}\}.$$

This is the set consisting of all polynomials (no restrictions on the degrees). Essentially the same argument we gave above for P_n shows that $P \subseteq F(-\infty, \infty)$

is a subspace. In fact, we also have that $P_n \subseteq P$ is a subspace of P. Furthermore, we can get really fancy and observe that we have an infinite sequence of subspaces

$$P_0 \subseteq P_1 \subseteq P_2 \subseteq P_3 \subseteq \ldots \subseteq P \subseteq F(-\infty, \infty).$$

2.2.2 The Subspace of Continuous Functions: $C(a, b)$

Let a, b be real numbers with $a < b$, or $a = -\infty, b = \infty$. Then define

$$C(a, b) := \{f \in F(a, b) \mid f \text{ is continuous}\}$$

(The 'C' stands for 'continuous'.) So $\sin(x), x^2 \in C(-1, 1)$, but the following piecewise function is not in $C(-1, 1)$:

$$f(x) = \begin{cases} x & x < 0 \\ 1 & x \geq 0 \end{cases}$$

$C(a, b)$ is a subspace of $F(a, b)$. This is immediate given the following theorem from calculus:

Theorem 5. *If f, g are continuous on an interval I and k is a real number, then*

$$f + g, \qquad kf$$

are both continuous on the same interval I.

For a proof see the section on continuity in [1][Rogawski].

2.2.3 The Subspace $C_0^2(a, b)$

Let a, b be real numbers with $a < b$. Then define

$$C_0^2(a, b) := \{f \in F(a, b) \mid f, f' \text{ both exist and are continuous, and } f(a) = f(b) = 0\}.$$

Exercise 9. *Prove that $C_0^2(a, b)$ is a subspace of $F(a, b)$. You should quote a theorem from calculus that is similar to Theorem 5 above.*

References

[1] J. Rogawski. *Calculus: Early Transcendentals.* W. H. Freeman. 2007.

[2] W. Wright, D. Zill. *Advanved Engineering Mathematics.* 4 ed. Jones and Bartlett. 2011.