S4D03/S6D03 2019/2020: Assignment Two

1. Let f be a real-valued measurable function on the probability space $(\Omega, \mathcal{F}, P)$. Assume that $f(\omega) \geq 1$ almost surely under P and $\int f(\omega) P(d\omega) = 1$. Show that $f(\omega) = 1$ almost surely under P.

2. Let $\{A_n\}_{n\geq 1}$ and $\{B_n\}_{n\geq 1}$ be two sequences of measurable sets in the measurable space $(\Omega, \mathcal{F})$. Set $C_n = A_n \cap B_n, D_n = A_n \cup B_n$.

(1) Show that

$$\left(\overline{\lim_{n\to\infty}} A_n\right) \cap \left(\overline{\lim_{n\to\infty}} B_n\right) \supset \overline{\lim_{n\to\infty}} C_n$$

and

$$\left(\underline{\lim_{n\to\infty}} A_n\right) \cup \left(\underline{\lim_{n\to\infty}} B_n\right) \subset \underline{\lim_{n\to\infty}} D_n.$$

(2) Show by example the two inclusions in (1) can be strict.

3. Consider the following two simple functions on a probability space $(\Omega, \mathcal{F}, P)$

$$f(\omega) = \sum_{i=1}^{3} a_i I_{A_i}(\omega),$$
$$g(\omega) = \sum_{j=1}^{4} b_j I_{B_j}(\omega).$$

Find $\int (f(\omega) + g(\omega))^2 P(d\omega)$.

4. Let $X_n, n \geq 2$ be a sequence of random variables such that

$$P\{X_n = 0\} = 1 - \frac{2}{n^2},$$
$$P\{X_n = n\} = P\{X_n = -n\} = \frac{1}{n^2}.$$

Show that $\{X_n\}_{n\geq 2}$ converges to 0 almost surely.

Due date: 3:30pm October 3, 2019 in class.