**S4D03/S6D03 2019/2020: Assignment Three**

1. Let $X_n$ be a binomial random variable with parameters $n$ and $p_n$, and $X$ be a Poisson random variable with parameter $\lambda > 0$. Assume that

$$\lim_{n\to\infty} np_n = \lambda.$$

Show that $X_n$ converges in distribution to $X$ as $n$ tends to infinity.

**Proof:** For any $k = 0, 1, \ldots$, we can choose $n$ large enough so that $k \leq n$. By direct calculation, we have

$$\begin{aligned} P(X_n = k) &= \binom{n}{k} p_n^k (1-p_n)^{n-k} \\ &= \frac{n!}{(n-k)!k!} \frac{(np_n)^k}{n^k} (1 - \frac{np_n}{n})^n (1-p_n)^{-k} \\ &= \frac{(np_n)^k}{k!} (1 - \frac{np_n}{n})^n \frac{n(n-1)\cdots(n-k+1)}{n^k} (1-p_n)^{-k} \\ &\longrightarrow \frac{\lambda^k}{k!} e^{-\lambda} = P(Y = k) \end{aligned}$$

where $Y$ is a Poisson random variable with parameter $\lambda$.

2. Let $X$ be a random variable with cumulative distribution function $F(\cdot)$ and

$$X_n = \frac{n}{n + \sqrt{n}} X.$$

Show that $X_n$ converges to $X$ in distribution.

**Proof:** For any continuity point $a$ of $F(\cdot)$ we have

$$\begin{aligned} F_{X_n}(a) &= P(X_n \leq a) = P(X \leq \frac{n+\sqrt{n}}{n} a) \\ &= F_X(\frac{n+\sqrt{n}}{n} a) \to F_X(a) \quad \text{as } n \to \infty. \end{aligned}$$

3. Consider a sequence of non-negative random variables $\{Y_n : n \geq 1\}$ satisfying

$$\sum_{n=2}^{\infty} P\{Y_n > \log n\} < \infty.$$

Show that

$$\overline{\lim_{n\to\infty}} \frac{Y_n}{\log n} \leq 1.$$

**Proof:** For any $n \geq 1$, let $B_n = \{\omega : Y_n(\omega) > \log n\} = \{\omega : \frac{Y_n}{\log n} > 1\}$. Since

$$\sum_{n=1}^{\infty} P(B_n) < \infty,$$

it follows from the Borel-Cantelli lemma that

$$P(\cap_{n=1}^{\infty} \cup_{m=n}^{\infty} B_m) = P(\overline{\lim_{n\to\infty}} B_n) = 0.$$

On the other hand, for any integer $r \geq 1$ set

$$A_r = \{\omega : \overline{\lim_{n\to\infty}} \frac{Y_n}{\log n} \geq 1 + \frac{1}{r}\}.$$

Noting that

$$\overline{\lim_{n\to\infty}} \frac{Y_n}{\log n} = \inf_n \sup_{m\geq n} \{\frac{Y_m}{\log m}\},$$

it follows that

$$\begin{aligned} A_r &= \{\omega : \inf_n \sup_{m\geq n} \{\frac{Y_m}{\log m} \geq 1 + \frac{1}{r}\} \\ &= \{\omega : \sup_{m\geq n} \{\frac{Y_m}{\log m} \geq 1 + \frac{1}{r} \text{ for all } n\} \\ &= \cap_{n=1}^{\infty} \cup_{m=n}^{\infty} \{\omega : \frac{Y_m}{\log m} \geq 1 + \frac{1}{r}\} \subset \overline{\lim_{n\to\infty}} B_n. \end{aligned}$$

Thus $P(A_r) = 0$. Since

$$A = \{\omega : \overline{\lim_{n\to\infty}} \frac{Y_n}{\log n} > 1\} = \cup_{r=1}^{\infty} A_r,$$

it follows that $P(A) = 0$ and

$$P(A^c) = P(\overline{\lim_{n\to\infty}} \frac{Y_n}{\log n} \leq 1) = 1.$$

4. Let $X_1, X_2, \ldots$ be iid exponential random variables with parameter $c > 0$. Set

$$M_n = \max\{X_1, \ldots, X_n\}, \quad b_n = c^{-1} \log n.$$

Let $M$ be a random variable with cumulative distribution function $e^{-e^{-cx}}$. Show that $M_n - b_n$ converges to $M$ in distribution as $n$ tends to infinity.

**Proof:**

$$\begin{aligned} P(M_n - b_n \leq x) &= P(M_n \leq x + b_n) \\ &= \Big(P(X_1 \leq x + b_n)\Big)^n \\ &= \Big(1 - e^{-c(x+b_n)}\Big)^n \\ &= (1 - \frac{e^{-cx}}{n})^n \to e^{-e^{-cx}} \\ &= P(M \leq x). \end{aligned}$$