S4D03/S6D03 2019/2020: Assignment Four

1. Let X, Y be two independent Poisson random variables with corresponding parameters $\lambda_1 > 0$ and $\lambda_2 > 0$. Find the characteristic function of $X + Y$ and identify its distribution.

2. For each $n \geq 1$, let X_n be a random variable with distribution function

$$F_n(x) = \begin{cases} 0, & x < 0 \\ x - \frac{\sin(2n\pi x)}{2n\pi}, & 0 \leq x < 1 \\ 1, & \text{else.} \end{cases}$$

(a) Show that $F_n(x)$ is indeed a distribution function.
(b) Show that X_n has a density function.
(c) Show that $F_n(x)$ converges in distribution to the uniform random variable X over $[0, 1]$ as n tends to infinity.
(d) Show that the density function of X_n does not converge to the density function of X.

3. Let $X_1, X_2, \ldots$ be i.i.d. with common finite mean -2 and variance 1. Show that

$$\frac{1}{n^2} \sum_{i,j=1,\ i \neq j}^{n} X_i X_j$$

converges almost surely to 4.

4. Show that for $x \geq 0$,

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{k:|2k-n| \leq \sqrt{n}x} \binom{n}{k} = \int_{-x}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} d\,u,$$

$$\lim_{n\to\infty} \sum_{k:|k-n| \leq \sqrt{n}x} \frac{n^k}{k!} e^{-n} = \int_{-x}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} d\,u.$$

Due date: 3:30pm on November 11, 2019.