S4D03/S6D03 2019/2020: Assignment Four

1. Let X, Y be two independent Poisson random variables with corresponding parameters $\lambda_1 > 0$ and $\lambda_2 > 0$. Find the characteristic function of $X + Y$ and identify its distribution.

Proof:

$$\begin{aligned}
\phi_{X+Y}(t) &= E[e^{it(X+Y)}] \\
&= E[e^{itX}]E[e^{itY}] \\
&= \sum_{k=0}^{\infty} e^{itk}\frac{\lambda_1^k}{k!}e^{-\lambda_1}\sum_{j=0}^{\infty} e^{itj}\frac{\lambda_2^k}{k!}e^{-\lambda_2} \\
&= e^{\lambda_1(e^{it}-1)}e^{\lambda_2(e^{it}-1)} \\
&= e^{(\lambda_1+\lambda_2)(e^{it}-1)}
\end{aligned}$$

which implies that $X + Y$ is a Poisson random variable with parameter $\lambda_1 + \lambda_2$.

2. For each $n \geq 1$, let X_n be a random variable with distribution function

$$F_n(x) = \begin{cases} 0, & x < 0 \\ x - \frac{\sin(2n\pi x)}{2n\pi}, & 0 \leq x < 1 \\ 1, & \text{else.} \end{cases}$$

(a) Show that $F_n(x)$ is indeed a distribution function.
(b) Show that X_n has a density function.
(c) Show that $F_n(x)$ converges in distribution to the uniform random variable X over $[0, 1]$ as n tends to infinity.
(d) Show that the density function of X_n does not converge to the density function of X.

Proof: (a) By definition, we have that $F_n(x)$ is continuous in x and

$$F_n(-\infty) = 0, \; F_n(+\infty) = 1.$$

The fact that

$$\frac{d\, F_n(x)}{d\, x} \geq 0$$

implies that $F_n(x)$ is non-decreasing in x. This $F_n(x)$ is a distribution function.

(b) Let

$$f_n(x) = \begin{cases} 0, & x < 0 \text{or } \geq 1 \\ 1 - \cos(2n\pi x), & 0 \leq x < 1 \end{cases}$$

Then it is clear that $f_n(x) geq 0$ and

$$\int_{-\infty}^{+\infty} f_n(x) d\, x = \int_0^1 (1 - \cos(2\pi n x)) d\, x = 1.$$

Thus $f_n(x)$ is a probability density function. By direct calculation we have

$$\frac{d\, F_n(x)}{d\, x} = f_n(x).$$

Therefore $f_n(x)$ is actually the density function of X_n.

(c) The cdf of X is continuous and is given by

$$F(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x < 1 \\ 1, & \text{else.} \end{cases}$$

For any $x < 0$ or $x \geq 1$, we have $F_n(x) = F(x)$. For $0 \leq x < 1$, we have

$$\lim_{n \to \infty} F_n(x) = \begin{cases} 0, & x < 0 \\ x - \lim_{n \to \infty} \frac{\sin(2n\pi x)}{2n\pi}, & 0 \leq x < 1 \\ 1, & \text{else} \end{cases} \qquad = F(x).$$

(d) The pdf of X is

$$f(x) = \begin{cases} 0, & x < 0 \text{or } \geq 1 \\ 1, & 0 \leq x < 1 \end{cases}$$

Since $f_n(\frac{1}{2}) = 0 \text{or } 2$ while $f(\frac{1}{2}) = 1$, it follows that f_n does not converge to $f(x)$.

3. Let $X_1, X_2, \ldots$ be i.i.d. with common finite mean -2 and variance 1. Show that

$$\frac{1}{n^2} \sum_{i,j=1,\ i \neq j}^{n} X_i X_j$$

converges almost surely to 4.

Proof: Set

$$U_n = \frac{1}{n} \sum_{i=1}^{n} X_i,$$

$$V_n = \frac{1}{n} \sum_{i=1}^{n} X_i^2,$$

and

$$W_n = \frac{1}{n^2} \sum_{i,j=1,\ i \neq j}^{n} X_i X_j.$$

Then we have

$$W_n = U_n^2 - \frac{1}{n} V_n.$$

By the strong law of large numbers, we have

$$U_n^2 \longrightarrow 4, \quad \text{almost surely}$$

and

$$V_n \longrightarrow 5, \quad \text{almost surely}.$$

Thus we have the desired the result.

4. Show that for $x \geq 0$,

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{k:|2k-n|\leq\sqrt{n}x} \binom{n}{k} = \int_{-x}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} d\,u,$$
$$\lim_{n\to\infty} \sum_{k:|k-n|\leq\sqrt{n}x} \frac{n^k}{k!} e^{-n} = \int_{-x}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} d\,u.$$

Proof: Let $X_1, X_2, \ldots$ be i.i.d. with

$$P(X_1 = 1) = \frac{1}{2} = P(X_1 = 0)$$

and

$$S_n = \sum_{i=1}^{n} X_i.$$

Then we have $E[X] = \mu = \frac{1}{2}, Var[X] = \sigma^2 = \frac{1}{4}$ and

$$P(|\frac{S_n - n\mu}{\sqrt{n\sigma^2}}| \leq x) = \frac{1}{2^n} \sum_{k:|2k-n|\leq\sqrt{n}x} \binom{n}{k}.$$

By Central Limit Theorem,

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{k:|2k-n|\leq\sqrt{n}x} \binom{n}{k} = \int_{-x}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} d\,u.$$

Next let Y_n be a Poisson random variable with parameter n.Then

$$P(\frac{Y_n - n}{\sqrt{n}} \leq x) = \sum_{k:|k-n|\leq\sqrt{n}x} \frac{n^k}{k!} e^{-n}.$$

By direct calculation,

$$\begin{aligned} E[e^{it(\frac{Y_n-n}{\sqrt{n}})}] &= e^{-it\sqrt{n}} e^{n(e^{\frac{it}{\sqrt{n}}}-1)} \\ &= e^{-\frac{t^2}{2}+o(1)}. \end{aligned}$$

By the continuity theorem, we get

$$\sum_{k:|k-n|\leq\sqrt{n}x} \frac{n^k}{k!} e^{-n} = \int_{-x}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} d\,u.$$